MATHEMATICS-II

B.E. II-SEMESTER (Common to All Branches)

(Group-A & Group-B)

Contents

Syllabus

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UNIT-2

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UNIT-3

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UNIT 1

MATRICES

PART-A
SHORT QUESTIONS WITH SOLUTIONS

Q1. Define elementary matrix.

Answer :

A matrix obtained by performing an elementary row or column operation on an identity matrix is known as elementary matrix or *E*-matrix.

Q2. Define rank of a matrix. Give an example of a 2 × 3 matrix whose rank is 2.

Answer : **(June/July-17, Q1 | June-14, Q1)**

Rank of a matrix

Let '*A*' represents a non-zero matrix. A number '*r*' is said to be the rank of matrix '*A*', if

(i) Every $(r+1)^{th}$ order minor of *A* is zero.

(ii) Atleast one minor of order '*r*' which is non-zero.

Example

$$A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}$$

$$\begin{aligned} \det A &= -1(18-1) - 0(9+5) + 6(3+30) \\ &= -17 - 0 + 198 \\ &= 181 \neq 0 \end{aligned}$$

$\because$ Minor of order $3 \neq 0$

$\therefore \quad \rho(A) = 3$

Example of a 2 × 3 matrix

Consider a 2 × 3 matrix

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

The determinant of submatrix,

$$\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$$

$\therefore$ The rank of *A* is 2.

Q3. What is meant by Echelon form of a matrix?

Answer :

A $m \times n$ matrix is said to be in Echelon form if,

- All zero rows or any zero row (if any) occurs below the non-zero row.
- The number of zeros before the first non-zero element in the rows are in the increasing order.
- The first non-zero element in every row is unity (optional).

Examples

(i) $$\begin{bmatrix} 1 & 2 & 4 & 0 & 3 \\ 0 & 1 & 3 & 5 & 7 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(ii) $$\begin{bmatrix} 1 & 0 & 4 & 13 \\ 0 & 1 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Q4. Convert the matrix into echelon form $\begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 6 & 2 & 4 \end{bmatrix}$

Answer : Model Paper-1, Q1

Given matrix is,

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 6 & 2 & 4 \end{bmatrix}$$

$R_2 \to 3R_2 - 2R_1$

$R_3 \to R_3 - 2R_1$

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & -2 & 2 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The number of non-zero rows in matrix $A = 2$

$\therefore$ Rank of A, $\rho(A) = 2$.

Q5. Find the rank of a matrix A = $\begin{bmatrix} -1 & 2 & 1 & 8 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 7 \end{bmatrix}$.

Answer :

Given matrix is,

$$A = \begin{bmatrix} -1 & 2 & 1 & 8 \\ 2 & 1 & -1 & 0 \\ 3 & 2 & 1 & 7 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1, R_3 \to R_3 + 3R_1$

$$\Rightarrow A = \begin{bmatrix} -1 & 2 & 1 & 8 \\ 0 & 5 & 1 & 16 \\ 0 & 8 & 4 & 31 \end{bmatrix}$$

$R_3 \to 5R_3 - 8R_2$

$$\Rightarrow A = \begin{bmatrix} -1 & 2 & 1 & 8 \\ 0 & 5 & 1 & 16 \\ 0 & 0 & 12 & 27 \end{bmatrix}$$

$R_3 \to \dfrac{R_3}{3}$

$$\Rightarrow A = \begin{bmatrix} -1 & 2 & 1 & 8 \\ 0 & 5 & 1 & 16 \\ 0 & 0 & 4 & 9 \end{bmatrix}$$

The number of non-zero rows in matrix $A = 3$.

$\therefore$ Rank of A, $\rho(A) = 3$.

Q6. Determine the rank of a matrix A = $\begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$

Answer : Model Paper-2, Q1

Given matrix is,

$$A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 2 & 3 & 7 & 5 \\ 2 & 5 & 11 & 6 \end{bmatrix}$$

$R_3 \to R_3 - R_1, R_4 \to R_4 - R_1$

$$A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 4 & 4 & 1 \\ 0 & 6 & 8 & 2 \end{bmatrix}$$

$R_4 \to \dfrac{R_4}{2}, R_3 \to 3R_3 - 4R_2$

$$A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 0 & -4 & -1 \\ 0 & 3 & 4 & 1 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$A = \begin{bmatrix} 2 & -1 & 3 & 4 \\ 0 & 3 & 4 & 1 \\ 0 & 0 & -4 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Number of non-zero rows = 3

$\therefore$ Rank of matrix = 3.

Q7. Determine the rank of a matrix A = $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

Answer : Model Paper-3, Q1

Given matrix is,

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1, R_4 \to R_4 - 6R_1$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_4 \to R_4 - R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Number of non-zero rows = 3.

∴ The rank of matrix is 3.

Q8. When does a non-homogeneous system consistent?

Answer :

A non-homogeneous system is represented in matrix form as $AX = B$

Where, $B \neq 0$.

A system $AX = B$ is said to be consistent, if and only if rank of A = rank of $[A/B]$

Where, $[A/B]$ is the augmented matrix.

Q9. Show that the equations: x – 4y + 7z = 14, 3x + 8y – 2z = 13, 7x – 8y + 26z = 5 are not consistent.

Answer :

Given,

$x - 4y + 7z = 14$

$3x + 8y - 2z = 13$

$7x - 8y + 26z = 5$

Writing the given equations in matrix form, $AX = B$

$$\begin{bmatrix} 1 & -4 & 7 \\ 3 & 8 & -2 \\ 7 & -8 & 26 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 14 \\ 13 \\ 5 \end{bmatrix}$$

The augmented matrix, $C = [A \mid B]$

$$C = \left[\begin{array}{ccc|c} 1 & -4 & 7 & 14 \\ 3 & 8 & -2 & 13 \\ 7 & -8 & 26 & 5 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 7R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & -4 & 7 & 14 \\ 0 & 20 & -23 & -29 \\ 0 & 20 & -23 & -93 \end{array}\right] \quad R_2(1/20)$$

$$= \left[\begin{array}{ccc|c} 1 & -4 & 7 & 14 \\ 0 & 1 & -23/20 & -29/20 \\ 0 & 20 & -23 & -93 \end{array}\right]$$

$R_1 \to R_1 + 4R_2$

$R_3 \to R_3 - 20R_2$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 12/5 & 41/5 \\ 0 & 1 & -23/20 & -29/20 \\ 0 & 0 & 0 & -64 \end{array}\right]$$

Here,

$\rho(C) = 3$ and $\rho(A) = 2$

i.e., Rank of $A \neq$ Rank of C

∴ The given equations are not consistent.

Q10. Prove that two vectors are linearly dependent if one of them is a scalar multiple of the other.

Answer :

Let α and β represents two linearly dependent vectors of the vector space V. There exists scalars $a, b \neq 0$ such that,

$a\alpha + b\beta = 0$

For $a \neq 0$, $a\alpha = -b\beta$

$\Rightarrow \quad \alpha = \left(\frac{-b}{a}\right)\beta$

∴ α is a scalar multiple of β

For $b \neq 0$, $b\beta = -a\alpha$

$\Rightarrow \quad \beta = \left(\frac{-a}{b}\right)\alpha$

∴ β is a scalar multiple of α.

Thus, it can be observed that if two vectors are linearly dependent, then one of the vector is a scalar multiple of the other.

Q11. If F is the field of real numbers, prove that the vectors (a_1, a_2) and (b_1, b_2) in V_2(F) are linearly dependent iff $a_1b_2 - a_2b_1 = 0$.

Answer :

Given vectors are,

$(a_1, a_2), (b_1, b_2)$

Let, $x, y \in F$

$\therefore \quad x(a_1, a_2) + y(b_1, b_2) = (0, 0)$

$\Rightarrow \quad (xa_1 + yb_1, xa_2 + yb_2) = (0, 0)$

$\Rightarrow \quad xa_1 + yb_1 = 0$ and $xa_2 + yb_2 = 0$

The necessary and sufficient condition for the above equations to have a non-zero solution is,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$$

$\Rightarrow \quad a_1b_2 - a_2b_1 = 0$

Hence, the given vectors are linearly dependent iff $a_1b_2 - a_2b_1 = 0$.

Q12. Show that the vectors (2, 1, 4), (1, –1, 2), (3, 1, –2) form the basis of R^3.

Answer :

Given vectors are,

(2, 1, 4), (1, –1, 2), (3, 1, –2)

Let, S = {(2, 1, 4), (1, –1, 2), (3, 1, –2)}

For 'S' to form the basis of R^3, $|S| \neq 0$ i.e., S should be linearly independent.

$$|S| = \begin{vmatrix} 2 & 1 & 4 \\ 1 & -1 & 2 \\ 3 & 1 & -2 \end{vmatrix}$$

$= 2(2 - 2) - (-2 - 6) + 4(1 + 3)$

$= 0 + 8 + 16$

$= 24 \neq 0$

$\therefore \quad |S| \neq 0$

The set 'S' is linearly independent.

Hence 'S' forms a basis of R^3.

Q13. Define linear transformation.

Answer : Dec.-15, Q9

Linear Transformation

If $U(F)$ and $V(F)$ represent two vector spaces in the field F, then linear transformation from U into V is a function T from U into V such that,

$T(a\alpha + b\beta) = aT(\alpha) + bT(\beta)$ ... (1)

$\forall$ α, β in U and a, b in F

Equation (1) refers to the linearity property.

Q14. Define the Eigen value and Eigen vector.

OR

Define,

(i) Characteristic roots and

(ii) Characteristic vectors of a square matrix.

Answer :

(i) Characteristic Root or Eigen Values or Latent Root

The roots of the characteristic equation $|A - \lambda I| = 0$ are called characteristic roots or latent roots or eigen values and are represented as $\lambda_1, \lambda_2 \ldots \lambda_n$.

(ii) Characteristic Vector or Eigen Vector or Latent Vector of a Square Matrix

Let $A = [a_{ij}]$ represents an $n \times n$ matrix. A non-zero vector X is said to be characteristic vector or latent vector or an eigen vector of a matrix A, if there exists a scalar λ such that $AX = \lambda X$.

Q15. If 3 and 5 are two eigen values of the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ then find its third eigen value and hence |A|.

Answer :

Given matrix is,

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

And eigen values, $\lambda_1 = 3$, $\lambda_2 = 5$

Sum of eigen values = Trace of matrix

$$= \sum_{i=1}^{n} a_{ii}$$

$\Rightarrow \quad \lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$

$\Rightarrow \quad 3 + 5 + \lambda_3 = 8 + 7 + 3$

$\Rightarrow \quad \lambda_3 = 18 - 8$

$\therefore \quad \lambda_3 = 10$

Determinant of matrix A = Product of eigen values

$\Rightarrow \quad |A| = \lambda_1\lambda_2\lambda_3$

$\Rightarrow \quad |A| = (3)(5)(10)$

$\therefore \quad |A| = 150$

Q16. Find the eigen values of the matrix $\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$

Answer : Model Paper-1, Q2

$$\text{Let, } A = \begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$$

The characteristic equation of A is given by,

$$|A - \lambda I| = 0$$

$$\Rightarrow \left|\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right| = 0$$

$$\Rightarrow \left|\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right| = 0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(4 - \lambda)((3 - \lambda)(1 - \lambda) - (2)(4)) - 2(-5(1 - \lambda)$
$- (2)(-2)) - 2(-5 \times 4 - (-2)(3 - \lambda)) = 0$

$\Rightarrow$ $(4 - \lambda)(3 - 4\lambda + \lambda^2 - 8) - 2(-5 + 5\lambda + 4)$
$- 2(-20 + 6 - 2\lambda) = 0$

$\Rightarrow$ $(4 - \lambda)(\lambda^2 - 4\lambda - 5) - 2(5\lambda - 1) - 2(-2\lambda - 14) = 0$

$\Rightarrow$ $4\lambda^2 - 16\lambda - 20 - \lambda^3 + 4\lambda^2 + 5\lambda - 10\lambda + 2$
$+ 4\lambda + 28 = 0$

$\Rightarrow$ $-\lambda^3 + 8\lambda^2 - 17\lambda - 10 = 0$

$\Rightarrow$ $\lambda^3 - 8\lambda^2 + 17\lambda - 10 = 0$

$\Rightarrow$ $\lambda = 5, 2, 1$

$\therefore$ 5, 2, 1 are the required eigen values.

Q17. Find the Eigen values of the following system
$8x - 4y = \lambda x$
$2x + 2y = \lambda y$

Answer :

Given equations are,

$$8x - 4y = \lambda x$$
$$2x + 2y = \lambda y$$

The above equations can be re-written as,

$$(8 - \lambda)x - 4y = 0 \quad \text{... (1)}$$
$$2x + (2 - \lambda)y = 0 \quad \text{... (2)}$$

Equations (1) and (2) can be written in matrix form as,

$$A = \begin{bmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{bmatrix}$$

The characteristic equation of matrix A is obtained by,

$$|A| = 0$$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -4 \\ 2 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(8 - \lambda)(2 - \lambda) - (-4)(2) = 0$

$\Rightarrow$ $16 - 8\lambda - 2\lambda + \lambda^2 + 8 = 0$

$\Rightarrow$ $\lambda^2 - 10\lambda + 24 = 0$

$\Rightarrow$ $\lambda^2 - 6\lambda - 4\lambda + 24 = 0$

$\Rightarrow$ $\lambda(\lambda - 6) - 4(\lambda - 6) = 0$

$\Rightarrow$ $(\lambda - 4)(\lambda - 6) = 0 \Rightarrow \lambda = 4, 6$

$\therefore$ Eigen values of given equations are, $\lambda = 4, 6$

Q18. Prove that the product of the eigen values is equal to the determinant of the matrix.

Answer :

Let, $A_{n\times n}$ be any square matrix such that $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are the eigen values. Then, characteristic equation is,

$$|A - \lambda I| = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2) \ldots\ldots (\lambda - \lambda_n) = 0$$

Substituting $\lambda = 0$ in the above equation,

$\Rightarrow$ $|A| = (-1)^n(-\lambda_1)(-\lambda_2)(-\lambda_3) \ldots\ldots (-\lambda_n)$

$\Rightarrow$ $|A| = (-1)^n (-1)^n \lambda_1 \lambda_2 \ldots. \lambda_n$

$= (-1)^{2n}\lambda_1 \lambda_2 \ldots\ldots \lambda_n$

$\Rightarrow$ $|A| = \lambda_1 \lambda_2 \ldots\ldots \lambda_n$

$\therefore$ The product of eigen values is equal to the determinant of the matrix.

Q19. If λ_1, λ_2, ..., λ_n are the eigen values of A, then prove that the eigen values of (A – kI) are (λ_1 – k), (λ_2 – k),..., (λ_n – k).

Answer :

Let, X be an eigen vector corresponding to the eigen value λ_1 of A. Then,

$$AX = \lambda_1 X$$

Adding $-kX$ on both sides.

$$AX - kX = \lambda_1 X - kX$$

$\Rightarrow$ $(A - kI)X = (\lambda_1 - k)X$

$\Rightarrow$ $(\lambda_1 - k)$ is the eigen value of $(A - kI)$.

Similarly $(\lambda_2 - k)$,... $(\lambda_n - k)$ are eigen values of $(A - kI)$.

Q20. If A and B are n rowed square matrices and if A is invertible, show that $A^{-1}B$ and BA^{-1} have same eigen values.

Answer :

Given that,

The matrix A is invertible,

$$A^{-1}B = B^{-1}(BA^{-1})B \qquad [\because BB^{-1} = 1]$$

Let, $A^{-1}B = P$, $BA^{-1} = Q$

$\Rightarrow$ $P = B^{-1}(Q)B$

$\Rightarrow$ $P - \lambda I = B^{-1}QB - \lambda I$

$= B^{-1}QB - \lambda B^{-1}B$

$= B^{-1}QB - B^{-1}\lambda B$

$= B^{-1}(QB - \lambda B)$

$\Rightarrow$ $P - \lambda I = B^{-1}(Q - \lambda I)B$

$\Rightarrow$ $|P - \lambda I| = |Q - \lambda I| \, |BB^{-1}|$

$\Rightarrow$ $|P - \lambda I| = |Q - \lambda I|$

Characteristic equation of P and Q are same.

$\therefore$ Eigen values of $A^{-1}B$ and BA^{-1} are same.

Q21. If λ be an eigen value of the non-singular matrix A, show that $\frac{|A|}{\lambda}$ is an eigen value of adj A.

Answer :

λ is an eigen value of the matrix A then there exists a non-zero matrix X such that,

$$AX = \lambda X$$

$$\Rightarrow (\text{adj } A)(AX) = (\text{adj } A)(\lambda X)$$

$$\Rightarrow (\text{adj } A) AX = \lambda(\text{adj } A)(X)$$

$$\Rightarrow |A| IX = \lambda (\text{adj } A) X$$

$$\Rightarrow |A| X = \lambda (\text{adj } A) X$$

$$\Rightarrow \frac{|A|}{\lambda} X = (\text{adj } A) X$$

$$\therefore (\text{adj } A) X = \frac{|A|}{\lambda} X$$

Where,

$\frac{|A|}{\lambda}$ is an eigen value of adj A.

Q22. State Cayley-Hamilton theorem.

Answer :

If A is a square matrix of order $n \times n$ then it must satisfy its own characteristic equation.

i.e., if $|A - \lambda I| = (-1)^n (\lambda^n + p_1 \lambda^{n-1} + p_2 \lambda^{n-2} + ... + p_{n-1} \lambda + p_n)$ is the characteristic equation of A then matrix equation must be,

$$A^n + p_1 A^{n-1} + p_2 A^{n-2} + ... + p_{n-1} A + p_n I = 0$$

Q23. Use Cayley-Hamilton theorem to find A^8 if $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

Answer :

Given matrix is,

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

The characteristic equation of matrix A is given as,

$$|A - \lambda I| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1 - \lambda)(-1 - \lambda) - 2(2) = 0$$

$$\Rightarrow -1 - \lambda + \lambda + \lambda^2 - 4 = 0$$

$$\Rightarrow \lambda^2 - 5 = 0$$

By Cayley–Hamilton theorem,

$$A^2 - 5I = 0$$

$$\Rightarrow A^2 = 5I$$

$$A^8 = (A^2)^4 = (5I)^4 = 5^4 I$$

$$= 625I$$

$$\therefore A^8 = 625I$$

Q24. Define canonical form of a matrix.

Answer :

Let A be a symmetric matrix corresponding to the quadratic form $Q = X^T A X$, and let D be the diagonal form of A, obtained by using the congruence operations. Then a non-singular matrix N is obtained such that,

$$D = N^T A N$$

Let,

$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ be a new set of variables described by the linear transformation.

$$X = NY$$

$$\Rightarrow X^T = (NY)^T = Y^T N^T$$

$$\Rightarrow Q = X^T A X = Y^T (N^T A N) Y = Y^T DY$$

$$= [y_1 \ y_2 ... y_n] \begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & c_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ y_n \end{bmatrix}$$

$$= c_1 y_1^2 + c_2 y_2^2 + \ldots + c_n y_n^2$$

This new form is simply a sum of squares, which is called a canonical form of Q.

Q25. Define rank, index and signature of a matrix.

Answer :

Rank

Rank of the matrix is equal to number of non-zero rows in a canonical form.

Index

Index of the matrix is equal to number of positive sign in the diagonal of matrix A.

Example

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$\therefore$ Rank, $r = 4$

Index, $s = 3$

Signature

Signature of the matrix is given by,

$$\text{Signature} = 2s - r = 2(3) - 4 = 2$$

$\therefore$ Signature = 2

Q26. Find the matrix of the quadratic form q = x² – 6xy + 3y².

Answer :

Given quadratic form is,

$q = x^2 - 6xy + 3y^2$

The matrix can be written as,

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore \quad A = \begin{bmatrix} 1 & -3 \\ -3 & 3 \end{bmatrix}$$

Q27. Write the quadratic form corresponding to the symmetric matrix $\begin{bmatrix} 1 & 0 & 4 \\ 0 & -2 & -1 \\ 4 & -1 & 3 \end{bmatrix}$.

Answer :

Given symmetric matrix is,

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -2 & -1 \\ 4 & -1 & 3 \end{bmatrix}$$

The quadratic form corresponding to the symmetric matrix A is given as X^TAX,

$$\text{Where, } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$X^TAX = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 4 \\ 0 & -2 & -1 \\ 4 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} 1 & 0 & 4 \\ 0 & -2 & -1 \\ 4 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} x + 0y + 4z \\ 0 - 2y - z \\ 4x - y + 3z \end{bmatrix}$$

$$= \begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} x + 4z \\ -2y - z \\ 4x - y + 3z \end{bmatrix}$$

$$= x(x + 4z) + y(-2y - z) + z(4x - y + 3z)$$

$$= x^2 + 4zx - 2y^2 - zy + 4zx - zy + 3z^2$$

$$= x^2 - 2y^2 + 3z^2 - 2yz + 8zx$$

$\therefore$ The quadratic form is, $x^2 - 2y^2 + 3z^2 - 2yz + 8zx$

Q28. Write the nature of $2y_1^2 + 4y_2^2 + 5y_3^2$.

Answer : Model Paper-2, Q2

Given quadratic form is,

$$2y_1^2 + 4y_2^2 + 5y_3^2$$

The above expression can be expressed in the following matrix form as,

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

The characteristic equation of matrix A is given by $|A - \lambda I| = 0$

$$\Rightarrow \left| \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2 - \lambda)(4 - \lambda)(5 - \lambda) = 0$$

$$\Rightarrow \lambda = 2, 4, 5$$

$\therefore$ The eigen values of matrix A are 2, 4 and 5.

As all the eigen values of matrix A are positive, the nature of quadratic form is positive definite.

Q29. Write the nature of $-3y_1^2 - 2y_2^2 - y_3^2$

Answer : Model Paper-3, Q2

Given quadratic form is,

$$-3y_1^2 - 2y_2^2 - y_3^2$$

The matrix of the given quadratic form is,

$$A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The characteristic equation is given as,

$$|A - \lambda I| = 0$$

$$\Rightarrow \left| \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} -3-\lambda & 0 & 0 \\ 0 & -2-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-3 - \lambda)((-2 - \lambda)(-1 - \lambda)) = 0$$

$$\Rightarrow (\lambda + 3)(\lambda + 2)(\lambda + 1) = 0$$

$$\Rightarrow \lambda = -3, -2, -1$$

Since all the eigen values of the matrix are negative, the nature of quadratic form is negative definite.

PART-B
ESSAY QUESTIONS WITH SOLUTIONS

1.1 RANK OF A MATRIX, ECHELON FORM, SYSTEM OF LINEAR EQUATIONS

Q30. Define elementary matrix and explain about row and column transformations.

Answer :

Elementary Matrix

A matrix obtained by performing an elementary row or column operation on an identity matrix is known as elementary matrix or E-matrix.

The operations known as elementary row operations (transformations) are defined as,

(i) The interchange of rows i.e., i^{th} row and j^{th} row.

It is denoted as $R_i \leftrightarrow R_j$ or R_{ij}

(ii) The addition of a constant multiple of one row with another row.

It is denoted as $R_i \rightarrow R_i + kR_j$ or $R_{ij}(k)$ or $R_i + kR_j$

(iii) Multiplying any row by a non-zero scalar (k).

It is represented as $R_i \rightarrow kR_i$ or $R_i(k)$ or kR_i

The column transformations are defined in a similar way but are denoted as 'C' instead of 'R'.

Properties

(a) Let A and B be matrix, then,

$\sigma(AB) = (\sigma A)B$ and $\sigma(AB) = A(\sigma B)$

Where, σ represents row or column transformation.

(b) The elementary row (column) transformations of a matrix A can be obtained by pre or post multiplying a matrix by respective E-matrices.

(c) The inverse of the matrix,

$$E_{ij}^{-1} = E_{ij} \text{ and}$$

$$[E_{ij}(k)]^{-1} = E_i\left(\frac{1}{k}\right) \quad ; \quad [E_{ij}(k)]^{-1} = E_{ij}(-k)$$

Q31. Reduce the matrix $\begin{bmatrix} 3 & 1 & 4 & 6 \\ 2 & 1 & 2 & 4 \\ 4 & 2 & 5 & 8 \\ 1 & 1 & 2 & 2 \end{bmatrix}$ to echelon form and find its rank.

Answer :

The given matrix is,

$$A = \begin{bmatrix} 3 & 1 & 4 & 6 \\ 2 & 1 & 2 & 4 \\ 4 & 2 & 5 & 8 \\ 1 & 1 & 2 & 2 \end{bmatrix}$$

$R_2 \rightarrow 3R_2 - 2R_1$

$R_3 \rightarrow 3R_3 - 4R_1$

$R_4 \rightarrow 3R_4 - R_1$

$$= \begin{bmatrix} 3 & 1 & 4 & 6 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 2 & 2 & 0 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$= \begin{bmatrix} 3 & 1 & 4 & 6 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$= \begin{bmatrix} 3 & 1 & 4 & 6 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$R_4 \to R_4 - R_3$

$$= \begin{bmatrix} 3 & 1 & 4 & 6 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is in Echelon form and the number of non-zero rows is 3.

$\therefore$ Rank(A) = 3.

Q32. Reduce the matrix A = $\begin{pmatrix} 1 & 4 & 5 & 2 \\ 2 & 8 & 6 & 7 \\ 3 & 5 & 2 & 1 \\ -1 & 2 & 3 & 0 \end{pmatrix}$ **to echelon form and hence find is rank.**

Answer :

Given matrix is,

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 8 & 6 & 7 \\ 3 & 5 & 2 & 1 \\ -1 & 2 & 3 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 + R_1$

$$= \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 0 & -4 & 3 \\ 0 & -7 & -13 & -5 \\ 0 & 6 & 8 & 2 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$= \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -13 & -5 \\ 0 & 0 & -4 & 3 \\ 0 & 6 & 8 & 2 \end{bmatrix}$$

$R_4 \to 7R_4 + 6R_2$

$$= \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -13 & -5 \\ 0 & 0 & -4 & 3 \\ 0 & 0 & -22 & -16 \end{bmatrix}$$

$R_4 \to 4R_4 - 22R_3$

$$= \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -13 & -5 \\ 0 & 0 & -4 & 3 \\ 0 & 0 & 0 & -130 \end{bmatrix}$$

The above matrix is in Echelon form where the number of non-zero rows are 4.

$\therefore$ The Rank of Matrix A is 4.

Q33. Give a detailed account on the solutions of linear systems.

Answer :

A linear equation is defined as an equation of the form $a_1x_1 + a_2x_2 + a_3x_3 \ldots a_nx_n = b$

Where,

$x_1, x_2 \ldots x_n \to$ Unknowns

$a_1, a_2 \ldots a_n, b \to$ Constants.

Let the system of 'm' linear equations in 'x' unknown be,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n &= b_2 \\ a_{i1}x_1 + a_{i2}x_2 + \ldots a_{in}x_n &= b_i \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots \quad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots a_{mn}x_n &= b_m \end{aligned}\right\} \quad \ldots (1)$$

The matrix form of equation (1) can be written as,

$$AX = B \quad \ldots (2)$$

Where,

$A = [a_{ij}]$

$X = (x_1, x_2 \ldots x_n)^T$

$B = (b_1, b_2 \ldots b_m)^T$

The matrix $[A/B]$ is referred to as the management matrix.

Consistency

❖ A system is said to be consistent if,

Rank of matrix = Rank of augmented matrix = Number of unknowns

i.e., the system has at least one solution

❖ A system is said to be inconsistent if,

rank of matrix ≠ rank of augmented matrix

i.e., the system does not have any solution.

Homogeneous Equation

- A system is said to be homogeneous if $B = 0$ [in equation (2)]
- A system is said to be non-homogeneous if $B \neq 0$ [in equation (2)]

Trivial

If $X = 0$ (i.e., $x_1 = 0$, $x_2 = 0$ ... $x_n = 0$), solution of a homogeneous system is said to be a trivial solution.

If $X \neq 0$, solution of a homogeneous system is said to be a non-trivial solution.

Q34. Find the values of a and b such that the equations x + y + z = 6, x + 2y + 3z = 10, x + 2y + az = b have (i) no solution, (ii) unique solution and (iii) inifinite solutions.

Answer : June/July-17, Q11(a)

Given system of equations are,

$$x + y + z = 6$$

$$x + 2y + 3z = 10$$

$$x + 2y + a\,z = b$$

The above system of equations can be written in matrix form as,

$$AX = B$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & a \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ b \end{bmatrix}$$

Augmented matrix is,

$$[A|B] = \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & a & b \end{bmatrix}$$

Reducing the augment matrix to Echelon form by applying elementary row operations, i.e.,

$$R_2 \to R_2 - R_1$$

$$R_3 \to R_3 - R_1$$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & a-1 & b-6 \end{bmatrix}$$

$$R_3 \to R_3 - R_2$$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & a-3 & b-10 \end{bmatrix}$$

The possible solutions are,

Case (i) When a = 3, b ≠ 10

Rank of $A = 2$

Rank of $[A \mid B] = 3$

$\Rightarrow$ Rank of $A \neq$ Rank of $[A \mid B]$

$\therefore$ The system has no solution.

Case (ii) When a ≠ 3, b may have any value

Rank of $A = 3$

Rank of $[A \mid B] = 3$

$\Rightarrow$ Rank of A = Rank of $[A \mid B] = 3$ = number of unknowns

$\therefore$ The system has unique solution.

Case (iii) When a = 3, b = 10

Rank of $A = 2$

Rank of $[A \mid B] = 2$

$\Rightarrow$ Rank of A = Rank of $[A \mid B] = 2 <$ Number of unknowns

$\therefore$ The system has infinite number of solutions.

Q35. Show that the system of equations is consistent 2x – y – z = 2, x + 2y + z = 2, 4x – 7y – 5z = 2 and solve.

Answer :

Given system of equations is,

$$2x - y - z = 2$$

$$x + 2y + z = 2$$

$$4x - 7y - 5z = 2$$

The system of equations can be written in matrix form as,

$$AX = B$$

Where,

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Augmented matrix is $[A : B]$

$$\begin{bmatrix} 2 & -1 & -1 : & 2 \\ 1 & 2 & 1 : & 2 \\ 4 & -7 & -5 : & 2 \end{bmatrix}$$

Reducing argumented matrix to Echelon form by applying elementary row operations,

$$R_1 \to R_1 - R_2$$

$$R_2 \to 2R_2 - R_1$$

$$R_3 \to R_3 - 4R_2$$

$$= \begin{bmatrix} 1 & -3 & -2 : & 0 \\ 0 & 5 & 3 : & 2 \\ 0 & -15 & -9 : & -6 \end{bmatrix}$$

$$R_3 \to R_3 - 3R_2$$

$$= \begin{bmatrix} 1 & -3 & -2 : & 0 \\ 0 & 5 & 3 : & 2 \\ 0 & 0 & 0 : & 0 \end{bmatrix}$$

Here,

Rank of A_1, $\rho(A) = 2$

Rank of $[A : B]$, $\rho(A : B) = 2$

$\therefore$ The given equations are consistent with infinite number of solutions as $\rho(A) = \rho[A : B] <$ Number of variables

From the reduced matrix $[A : B]$,

$$x - 3y - 2z = 0 \quad \text{... (1)}$$

$$5y + 3z = 2 \quad \text{... (2)}$$

Let, $z = k$, where k is any real number.

Substituting $z = k$ in equation (2),

$\Rightarrow$ $5y + 3k = 2$

$\Rightarrow$ $5y = 2 - 3k$

$\Rightarrow$ $y = \dfrac{2-3k}{5}$

Substituting z, y values in equation (1),

$$x - 3\left(\frac{2-3k}{5}\right) - 2k = 0$$

$\Rightarrow$ $x = \dfrac{6-9k}{5} + 2k$

$\Rightarrow$ $x = \dfrac{6-9k+10k}{5}$

$\Rightarrow$ $x = \dfrac{k+6}{5}$

$\therefore$ The solution set is,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \dfrac{k+6}{5} \\ \dfrac{2-3k}{5} \\ k \end{bmatrix}.$$

Q36. Test for consistency and solve,

5x + 3y + 7z = 4

3x + 26y + 2z = 9

7x + 2y + 10z = 5.

Answer :

Given system of equations is,

$$5x + 3y + 7z = 4$$

$$3x + 26y + 2z = 9$$

$$7x + 2y + 10z = 5$$

The system of equations can be written in matrix form as,

$$AX = B$$

Where,

$$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; B = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

The system is consistent if rank A = Rank of $[A \mid B]$.

Where, $[A \mid B]$ is the augmented matrix.

$$[A \mid B] = \begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix}$$

$R_3 \to 5R_3 - 7R_1$

$R_2 \to 5R_2 - 3R_1$

$$= \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & -11 & 1 & -3 \end{bmatrix}$$

$R_2 \to \dfrac{R_2}{11}$

$$= \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 11 & -1 & 3 \\ 0 & -11 & 1 & -3 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$= \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 11 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is in Echelon form,

Number of non-zero rows is 2

$\Rightarrow$ Rank of A = Rank of $[A, B] = 2 <$ number of variables

$\therefore$ The matrix is consistent with infinite number of solutions,

From the above matrix,

$$11y - z = 3 \quad \text{... (1)}$$

$$5x + 3y + 7z = 4 \quad \text{... (2)}$$

Let, $y = k$

Substituting $y = k$ in equation (1),

$$z = 11k - 3$$

Substituting y, z values in equation (2),

$\Rightarrow$ $5x + 3k + 7(11k - 3) = 4$

$\Rightarrow$ $5x + 3k + 77k - 21 = 4$

$\Rightarrow$ $5x = -80k + 25$

$x = -16k + 5$

$\therefore$ The solution is,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = k\begin{bmatrix} -16 \\ 1 \\ 11 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix}$$

Q37. Discuss the consistency of the system of equations

$$\begin{aligned}2x+3y+4z&=11\\x+5y+7z&=15\\3x+11y+13z&=25\end{aligned}$$

Answer :

Given equations are,

$2x + 3y + 4z = 11$

$x + 5y + 7z = 15$

$3x + 11y + 13z = 25$

The given system of equations can be written in matrix form as,

$AX = B$

$$\Rightarrow \begin{bmatrix}2 & 3 & 4\\1 & 5 & 7\\3 & 11 & 13\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}11\\15\\25\end{bmatrix}$$

The argumented matrix, $C = [A \mid B]$

$$= \left[\begin{array}{ccc|c}2 & 3 & 4 & 11\\1 & 5 & 7 & 15\\3 & 11 & 13 & 25\end{array}\right]$$

$R_2 \to 2R_2 - R_1$
$R_3 \to 2R_3 - 3R_1$

$$= \left[\begin{array}{ccc|c}2 & 3 & 4 & 11\\0 & 7 & 10 & 19\\0 & 13 & 14 & 17\end{array}\right]$$

$R_3 \to 7R_3 - 13R_2$

$$= \left[\begin{array}{ccc|c}2 & 3 & 4 & 11\\0 & 7 & 10 & 19\\0 & 0 & -32 & -128\end{array}\right]$$

$R_3 \to \dfrac{R_3}{-32}$

$$\Rightarrow \quad C = \left[\begin{array}{ccc|c}2 & 3 & 4 & 11\\0 & 7 & 10 & 19\\0 & 0 & 1 & 4\end{array}\right]$$

Rank of A, $\rho(A) = 3$

Rank of (AB), $\rho[A \mid B] = 3$

As $\rho(A) = \rho[A \mid B] = 3$,

The given equations are consistent and has unique solution.

$$\therefore \begin{bmatrix}2 & 3 & 4\\0 & 7 & 10\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}11\\19\\4\end{bmatrix}$$

$\Rightarrow \quad 2x + 3y + 4z = 11$ ... (1)

$\Rightarrow \quad 7y + 10z = 19$ ... (2)

$\Rightarrow \quad z = 4$ ... (3)

Substituting $z = 4$ in equation (2)

$7y + 10z = 19$

$\Rightarrow \quad 7y + 10(4) = 19$

$\Rightarrow \quad 7y + 40 = 19$

$\Rightarrow \quad y = \dfrac{19-40}{7}$

$\Rightarrow \quad y = -3$

Substituting $y = -3$, $z = 4$ in equation (1)

$2x + 3y + 4z = 11$

$\Rightarrow \quad 2x + 3(-3) + 4(4) = 11$

$\Rightarrow \quad 2x - 9 + 16 = 11$

$\Rightarrow \quad x = \dfrac{11-16+9}{2}$

$\Rightarrow \quad x = 2$

$\therefore$ The solution is, $x = 2$, $y = -3$, $z = 4$.

1.2 LINEARLY DEPENDENCE AND INDEPENDENCE OF VECTORS, LINEAR TRANSFORMATION, ORTHOGONAL TRANSFORMATION

Q38. What is linear dependence and linear independence of vectors? Give examples for each.

Answer :

Linear Dependence of Vectors

A set of n-vectors i.e., $x_1, x_2, ..., x_n$ is said to be linearly dependent, when the linear combination of $n-1$ vectors is equal to one of the vectors.

Example

$x_1 = (1, 0, 2, 5)$

$x_2 = (2, 1, 2, 1)$

$x_3 = (3, 2, 1, 0)$

$x_4 = (-1, -1, -1, 7)$

Linear Independence of Vectors

A set of n-vectors i.e., $x_1, x_2, ..., x_n$ is said to be linearly independent, when none of the n-vectors are linearly dependent.

Example

$x_1 = (1, 0, 0, 0)$

$x_2 = (0, 1, 0, 0)$

$x_3 = (0, 0, 1, 0)$

Q39. Prove that (a) Every non-empty subset of a linearly independent set of vectors is linearly independent (b) Every superset of a linearly dependent set of vectors is linearly dependent.

Answer :

Let, $S = \{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_m\}$ be a linearly independent set of vectors,

Proof :

(a) $S' = \{\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_p\}$ be a subset of S,

Where, $1 \le P \le m$

Consider, $a_1, a_2, a_3, \ldots\ldots, a_p$ be the scalars.

Then, $a_1 \alpha_1 + a_2 \alpha_2 + a_3 \alpha_3 + \ldots\ldots + a_m \alpha_m = 0$

$\Rightarrow \quad a_1 \alpha_1 + a_2 \alpha_2 + a_3 \alpha_3 + \ldots\ldots + a_p \alpha_p + 0.\alpha_{p+1} + 0.\alpha_{p+2} + \ldots\ldots + 0.\alpha_m = 0$ ($\because$ S is linearly independent)

$\Rightarrow \quad a_1 = 0, a_2 = 0, a_3 = 0, \ldots\ldots, a_p = 0,$

i.e., S' is linearly independent

$\Rightarrow \quad S' = \{\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_p\}$ is linearly independent

$\therefore$ Subset of linearly independent set of vectors is linearly independent.

(b) Let, $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta\}$ be a superset of S.

For S, there exists scalars $a_1, a_2, \ldots, a_n$ not equal to zero, such that,

$$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = 0$$

Similarly, for S',

$$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n + 0\beta_1 + 0\beta_2 + \ldots + 0\beta_m = 0$$

From above equation, it can be observed that the scalar coefficients are not zero, hence S is said to be linearly dependent.

Therefore, any superset of linearly dependent set of vectors is linearly dependent.

Q40. Show that the set of three vectors (1, 3, 2), (1, –7, –8), (2, 1, –1) of $V_3(R)$ is linearly dependent.

Answer :

Given vectors are,

(1, 3, 2), (1, –7, –8), (2, 1, –1)

Let, $S = \{(1, 3, 2), (1, -7, -8)\ (2, 1, -1)\}$

The given vectors are linearly dependent, if and only if $|S| = 0$

$$\text{i.e., } |S| = \begin{vmatrix} 1 & 3 & 2 \\ 1 & -7 & -8 \\ 2 & 1 & -1 \end{vmatrix}$$

$= 1(7 + 8) -3\ (-1 + 16) + 2\ (1 + 14)$

$= 1\ (15) -3(15) + 2(15)$

$= 15 - 45 + 30 = 0$

$\therefore \quad |S| = 0$

Hence, the given vectors are linearly dependent.

Q41. Show that the vectors (1, 2, 1), (2, 1, 0), (1, –1, 2) of $V_3(R)$ is linearly independent.

Answer :

Given vectors are,

(1, 2, 1), (2, 1, 0), (1, –1, 2)

Let, $S = \{(1, 2, 1), (2, 1, 0), (1, -1, 2)\}$

The given vectors are linearly independent, if and only if $|S| \neq 0$.

i.e., $|S| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & -1 & 2 \end{vmatrix}$

$= 1(2 + 0) - 2(4 - 0) + 1(-2 - 1)$

$= 1(2) - 2(4) + 1(-3)$

$= 2 - 8 - 3 = -9 \neq 0$

$\therefore \quad |S| \neq 0$

Hence, the given vectors are linearly independent.

Q42. Explain in detail about linear transformation.

Answer :

Let $P(x, y)$ be any point in the Cartesian plane then its coordinates (x, y) denotes the set of rectangular axes OX and OY.

Let (x', y') be the coordinates of P which denotes to axes OX', OY' i.e., the former axes are rotated through the angle θ.

It is given by,

$x' = x \cos\theta + y \sin\theta$

$y' = -x \sin\theta + y \cos\theta$

It can also be written as,

$x' = a_1x + b_1y$

$y' = a_2x + b_2y$

and its matrix notation is,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Such transformations are known as linear transformations in two dimensions.

Similarly, the transformations in three dimensions are,

$x' = l_1x + m_1y + n_1z$

$y' = l_2x + m_2y + n_2z$

$z' = l_3x + m_3y + n_3z$

The linear transformations for n variables is given by relation $Y = AX$

Where,

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 & \cdots & k_1 \\ a_2 & b_2 & c_2 & \cdots & k_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_n & b_n & c_n & \cdots & k_n \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

This transformation is called linear because it hold the linear relations i.e., $A(X_1 + X_2) = AX_1 + AX_2$ and $A(bX) = bAX$. The transformation is said to be singular if the transformation matrix A is singular.

The non-singular transformation is the inverse of the relation $Y = AX$ i.e., $X = A^{-1}Y$. It is also referred as regular transformation.

Q43. Explain in detail about orthogonal transformation.

Answer :

The transformation is said to be orthogonal transformation if the linear transformation $Y = AX$ transforms $y_1^2 + y_2^2 + ... + y_n^2$ into $x_1^2 + x_2^2 + ... + x_n^2$ and its matrix is called the orthogonal matrix.

The matrices $X'X$ and $Y'Y$ are,

$$X'X = [x_1 \ x_2 \ ... \ x_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1^2 + x_2^2 + ... + x_n^2$$

and $$Y'Y = [y_1 \ y_2 \ ... \ y_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = y_1^2 + y_2^2 + ... + y_n^2$$

If $Y = AX$ is orthogonal transformation then,

$X'X = Y'Y$

$= (AX)'(AX)$ $[\because Y = AX; \ Y' = (AX)']$

$= X'A'AX$

$= X'X$ $[\because A'A = I]$

Since $A^{-1}A = I$

Also $A'A = I$

Comparing $A^{-1}A = A'A$

$\Rightarrow \quad A^{-1} = A'$

$\Rightarrow \quad A^{-1} = A'$ is the orthogonal transformation.

$\therefore$ The square matrix A is orthogonal if $AA' = A'A = I$

Q44. Show that the transformation $y_1 = 2x_1 + x_2 + x_3$, $y_2 = x_1 + x_2 + 2x_3$, $y_3 = x_1 - 2x_3$ is regular. Also write its inverse transformation.

Answer :

Given linear transformation are,

$y_1 = 2x_1 + x_2 + x_3$

$y_2 = x_1 + x_2 + 2x_3$

$y_3 = x_1 - 2x_3$

The above transformations can be written as,

$$Y = AX$$

Where, $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Consider,

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{vmatrix}$$

$$= 2[1(-2) - 0] - 1[1(-2) - 2(1)] + 1[0 - 1]$$

$$= 2[-2] - 1[-2 - 2] + 1[-1]$$

$$= -4 + 4 - 1$$

$$= -1$$

$\therefore$ $|A| = -1$

$\Rightarrow$ Matrix A is non-singular

$\therefore$ The given transformation is regular.

The inverse transformation is given by,

$$X = A^{-1}Y$$

Where,

$$A^{-1} = \frac{\text{Adj}A}{|A|}$$

Cofactor of 2 = $(-1)^{1+1}\begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} = -2$

Cofactor of 1 = $(-1)^{1+2}\begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = (-1)[-2 - 2] = 4$

Cofactor of 1 = $(-1)^{1+3}\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = (1)[0 - 1] = -1$

Cofactor of 1 = $(-1)^{2+1}\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} = (-1)[-2 - 0] = 2$

Cofactor of 1 = $(-1)^{2+2}\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = [-4 - 1] = -5$

Cofactor of 2 = $(-1)^{2+3}\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = (-1)[0 - 1] = 1$

Cofactor of 1 = $(-1)^{3+1}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = [2 - 1] = 1$

Cofactor of 0 = $(-1)^{3+2}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = (-1)[4 - 1] = -3$

Cofactor of –2 = $(-1)^{3+3}\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = [2 - 1] = 1$

$\therefore$ Adj A = [Cofactor of A]^T = $\begin{bmatrix} -2 & 4 & -1 \\ 2 & -5 & 1 \\ 1 & -3 & 1 \end{bmatrix}^T$

$\Rightarrow$ $\text{Adj } A = \begin{bmatrix} -2 & 2 & 1 \\ 4 & -5 & -3 \\ -1 & 1 & 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} -2 & 2 & 1 \\ 4 & -5 & -3 \\ -1 & 1 & 1 \end{bmatrix}$$

$\therefore$ $A^{-1} = \begin{bmatrix} 2 & -2 & -1 \\ -4 & 5 & 3 \\ 1 & -1 & -1 \end{bmatrix}$

$\therefore$ The inverse transformations are,

$$x_1 = 2y_1 - 2y_2 - y_3$$

$$x_2 = -4y_1 + 5y_2 + 3y_3$$

$$x_3 = y_1 - y_2 - y_3$$

Q45. Prove that the following matrix is orthogonal.

$$\begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$$

Answer : **Model Paper-3, Q16(a)**

Let the given matrix be,

$$A = \begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$$

$$A' = \begin{bmatrix} -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix}$$

The matrix A is orthogonal if,

$$AA' = I$$

Consider,

$$AA' = \begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}\begin{bmatrix} -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{9}+\frac{1}{9}+\frac{4}{9} & \frac{-4}{9}+\frac{2}{9}+\frac{2}{9} & -\frac{2}{9}-\frac{2}{9}+\frac{4}{9} \\ -\frac{4}{9}+\frac{2}{9}+\frac{2}{9} & \frac{4}{9}+\frac{4}{9}+\frac{1}{9} & \frac{2}{9}-\frac{4}{9}+\frac{2}{9} \\ -\frac{2}{9}-\frac{2}{9}+\frac{4}{9} & \frac{2}{9}-\frac{4}{9}+\frac{2}{9} & \frac{1}{9}+\frac{4}{9}+\frac{4}{9} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\Rightarrow$ $AA' = I$

$\therefore$ The matrix A is orthogonal

1.3 EIGENVALUES, EIGENVECTORS, PROPERTIES OF EIGENVALUES, CAYLEY-HAMILTON THEOREM

Q46. Explain the procedure to find the eigen values and eigen vector for a given matrix.

Answer :

Step 1

Initially, let the given matrix be,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Step 2

The next step is to find the characteristic matrix i.e., $(A - \lambda I)$.

$$\Rightarrow A - \lambda I = \begin{pmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{pmatrix}$$

Step 3

The characteristic equation i.e., $|A - \lambda I| = 0$ is determined to obtain the characteristic roots.

$$\text{i.e., } |A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

The characteristic roots of A are referred to as eigen values.

Step 4

The eigen vector X with respect to the eigen value λ are obtained as,

$$\begin{pmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Q47. Find the eigen values and the corresponding eigen vectors of A = $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix}$.

Answer : **(Model Paper-3, Q11(a) | June-14, Q16(a))**

Given matrix is,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix}$$

The characteristic equation of matrix A is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 0 & 0 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 1 \\ 0 & 0 & 6-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(1 - \lambda)[(4 - \lambda)(6 - \lambda) - (1)(0)] - 2[0(6 - \lambda) - 1(0)] + 3[(0)(0) - (0)(4 - \lambda)] = 0$

$\Rightarrow$ $(1 - \lambda)[24 - 4\lambda - 6\lambda + \lambda^2] - 2[0 - 0] + 3[0 - 0] = 0$

$\Rightarrow$ $(1 - \lambda)[\lambda^2 - 4\lambda - 6\lambda + 24] = 0$

$\Rightarrow$ $(1 - \lambda)[(\lambda - 4)(\lambda - 6) = 0$

$\therefore$ $\lambda = 1, 4, 6$ are the eigen values of A

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the required eigen vector then, $[A - \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 1 \\ 0 & 0 & 6-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{... (1)}$$

Eigen Vector Corresponding to $\lambda = 1$

Substituting $\lambda = 1$ in equation (1),

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From the above matrix

$2x_2 + 3x_3 = 0$

$3x_2 + x_3 = 0$

$5x_3 = 0$

$\Rightarrow$ $x_3 = 0$

$\Rightarrow$ $x_2 = 0$

Let

$x_1 = \alpha$

Then, the eigen vector corresponding to $\lambda = 1$ is,

$$X_1 = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Eigen Vector Corresponding to $\lambda = 4$

Substituting $\lambda = 4$ in equation (1),

$$\begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & -3 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From the above matrix,

$$-3x_1 + 2x_2 + 3x_3 = 0$$

$$-3x_3 = 0$$

$\Rightarrow$ $2x_3 = 0$

$\Rightarrow$ $x_3 = 0$

$\therefore$ $-3x_1 + 2x_2 = 0$

$$x_1 = \frac{2}{3}x^2$$

Let,

$$x_2 = \beta$$

Then, $x_1 = \frac{2}{3}\beta$

$\Rightarrow$ $x_1 = \frac{2}{3}\beta$

$$x_2 = \beta$$

$$x_3 = 0$$

$\therefore$ The eigen vector corresponding to $\lambda = 4$ is,

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}\beta \\ \beta \\ 0 \end{bmatrix} = \beta\begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix}$$

Eigen Value Corresponding to $\lambda = 6$

Substituting $\lambda = 6$ in equation (1),

$$\begin{bmatrix} -5 & 2 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From the above matrix,

$$-5x_1 + 2x_2 + 3x_3 = 0$$

$$-x_2 + x_3 = 0$$

$$x_3 = 0$$

$\Rightarrow$ $x_2 = x_3$

$$-5x_1 + 2x_2 + 3x_2 = 0$$

$$-5x_1 + 5x_2 = 0$$

$$x_1 = x_2$$

$\therefore$ $x_1 = x_2,\ x_2 = x_2,\ x_3 = x_2$

Let

$$x_2 = \gamma$$

$$x_1 = \gamma,\ x_2 = \gamma, x_3 = \gamma$$

$\therefore$ $X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \gamma \\ \gamma \\ \gamma \end{bmatrix} = \gamma\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\therefore$ The eigen vectors for eigen values $\lambda = 1, 4, 6$ are,

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ X_2 = \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix}, X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Q48. Find the eigen value and the corresponding eigen vectors of the matrix A = $\begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$.

Answer :

Given matrix is,

$$A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

The characteristic equation of A is $|A - \lambda I| = 0$

$\Rightarrow$ $\left|\begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right| = 0$

$\Rightarrow$ $\left|\begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right| = 0$

$\Rightarrow$ $\begin{vmatrix} 3-\lambda & 1 & -1 \\ -2 & 1-\lambda & 2 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow$ $(3 - \lambda)[(1 - \lambda)(2 - \lambda) - 2] - 1[-2(2 - \lambda) - 0] - 1[-2 - 0] = 0$

$\Rightarrow$ $(3 - \lambda)[2 - \lambda - 2\lambda + \lambda^2 - 2] - 1[-4 + 2\lambda] - 1[-2] = 0$

$\Rightarrow$ $(3 - \lambda)[\lambda^2 - 3\lambda] + 4 - 2\lambda + 2 = 0$

$\Rightarrow$ $3\lambda^2 - 9\lambda - \lambda^3 + 3\lambda^2 + 6 - 2\lambda = 0$

$\Rightarrow$ $-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$

$\Rightarrow$ $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

$\lambda = 1$	1	−6	11	−6
	0	1	−5	6
	1	−5	6	0

$\Rightarrow$ $(\lambda - 1)(\lambda^2 - 5\lambda + 6) = 0$

$\Rightarrow$ $(\lambda - 1)(\lambda^2 - 3\lambda - 2\lambda + 6) = 0$

$\Rightarrow$ $(\lambda - 1)(\lambda - 3)(\lambda - 2) = 0$

$\therefore$ $\lambda = 1, 2, 3$

Let, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be an eigen vector corresponding to eigen value λ.

Then $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 3-\lambda & 1 & -1 \\ -2 & 1-\lambda & 2 \\ 0 & 1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{... (1)}$$

Eigen value corresponding to λ = 1

Substituting λ = 1 in equation (1),

$$\Rightarrow \begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The above matrix reduces to the following equation

$\Rightarrow \quad 2x + y - z = 0$

$-2x + 2z = 0$

$y + z = 0$

$-2x + 2z = 0$

$\Rightarrow \quad x = z$

$y + z = 0$

$\Rightarrow \quad y = -z$

Let, $x = c_1 \Rightarrow y = -c_1$ and $z = c_1$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_1 \\ -c_1 \\ c_1 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\therefore \quad X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Eigen vector corresponding to λ = 2

Substituting λ = 2 in equation (1),

$$\begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The above matrix reduces to the following equation

$x + y - z = 0$

$-2x - y + 2z = 0$

$y = 0$

$\Rightarrow \quad x + 0 - z = 0$

$\Rightarrow \quad x = z$

Let, $x = c_1 \Rightarrow \quad z = c_1$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \\ c_1 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \quad X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Eigen vector corresponding to λ = 3

Substituting λ = 3 in equation (1),

$$\begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The above matrix reduces to the following equation,

$\Rightarrow \quad y - z = 0$

$\Rightarrow \quad -2x - 2y + 2z = 0$

$\Rightarrow \quad y - z = 0$

$\Rightarrow y = z$

$-2x - 2y + 2y = 0$

$\Rightarrow \quad x = 0$

Let, $y = c_1$

$\Rightarrow \quad z = c_1$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ c_1 \\ c_1 \end{bmatrix} = c_1\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore \quad X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Hence, the eigen vectors of A corresponding to eigen values λ = 1, 2, 3 are

$$X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{and } X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Q49. Find the eigen values and the corresponding eigen vectors of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Answer :

Given matrix is,

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

The characteristic equation is given as,

$$|A - \lambda I| = 0$$

$$\Rightarrow \left|\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(1-\lambda)[(5-\lambda)(1-\lambda)-1] - 1[1-\lambda-3] + 3[1-15+3\lambda] = 0$

$\Rightarrow$ $(1-\lambda)(4-6\lambda+\lambda^2) + 2 + \lambda - 42 + 9\lambda = 0$

$\Rightarrow$ $-\lambda^3 + 7\lambda^2 - 36 = 0$

$\Rightarrow$ $\lambda^3 - 7\lambda^2 + 36 = 0$

By inspection $\lambda = -2$ satisfies the equation,

$$\lambda = -2 \left| \begin{array}{cccc} 1 & -7 & 0 & 36 \\ 0 & -2 & 18 & -36 \\ \hline 1 & -9 & 18 & 0 \end{array} \right.$$

$\Rightarrow$ $(\lambda+2)(\lambda^2 - 9\lambda + 18) = 0$

$\Rightarrow$ $(\lambda+2)(\lambda-3)(\lambda-6) = 0$

Thus the eigen values of A are $\lambda = -2$, 3 and 6.

Let, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be an eigen vector corresponding to eigen values of λ.

Then, $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{... (1)}$$

Eigen Vector Corresponding to $\lambda = -2$

Substituting $\lambda = -2$ in equation (1),

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$3x + y + 3z = 0 \quad \text{... (2)}$$

$$x + 7y + z = 0 \quad \text{... (3)}$$

$$3x + y + 3z = 0 \quad \text{... (4)}$$

Equations (2) and (4) are same,

Considering equations (2) and (3),

$$\frac{x}{1-21} = \frac{y}{3-3} = \frac{z}{21-1}$$

$$\frac{x}{-20} = \frac{y}{0} = \frac{z}{20}$$

$\therefore$ The eigen vector for eigen value $\lambda = -2$ is, $(-1, 0, 1)$.

Eigen Vector Corresponding to $\lambda = 3$

Substituting $\lambda = 3$ in equation (1),

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & 1 & 3 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & -5 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which reduces to the equations,

$$x + 2y + z = 0 \quad \text{... (5)}$$

$$5y + 5z = 0$$

$\Rightarrow$ $y + z = 0 \Rightarrow z = -y$

Let,

$z = k$ then $y = -k$

Substituting the values in equation (5),

$x = k$

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ -k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Thus the eigen vector for eigen value $\lambda = 3$ is, $(1, -1, 1)$.

Eigen Vector Corresponding to $\lambda = 6$

Substituting $\lambda = 6$ in equation (1),

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which reduces to the equations,

$$-5x + y + 3z = 0 \quad \text{... (6)}$$

$$x - y + z = 0 \quad \text{... (7)}$$

$$3x + y - 5z = 0$$

Solving equations (6) and (7),

$$\begin{array}{r} -5x + y + 3z = 0 \\ x - y + z = 0 \\ \hline -4x + 4z = 0 \end{array}$$

$\therefore \quad -x + z = 0 \Rightarrow x = z$

Let, $x = k$ then $z = k$

Substituting the above values in equation (7),

$y = 2k$

$$X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k \\ 2k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Thus the eigen vector for eigen value $\lambda = 6$ is (1, 2, 1).

Q50. Prove that the eigen vectors corresponding to two different eigen values are linearly independent.

Answer :

Let, X_1 and X_2 be any two eigen vectors for any given matrix A for two distinct eigen values λ_1 and λ_2.

i.e., $AX_1 = \lambda_1 X_1$, $AX_2 = \lambda_2 X_2$

Let, $a_1 X_1 + a_2 X_2 = 0$ ... (1)

Multiplying equation (1) by matrix A,

$\Rightarrow \quad A(a_1X_1 + a_2X_2) = 0$

$\Rightarrow \quad Aa_1X_1 + Aa_2X_2 = 0$

$\Rightarrow \quad a_1(AX_1) + a_2(AX_2) = 0$

$\Rightarrow \quad a_1(\lambda_1X_1) + a_2(\lambda_2 X_2) = 0$

$\Rightarrow \quad \lambda_1 a_1 X_1 + \lambda_2 a_2 X_2 = 0$ ... (2)

From equation (1),

$a_1X_1 = -a_2X_2$

Substituting the above value in equation (2),

$\lambda_1 (-a_2X_2) + \lambda_2 a_2 X_2 = 0$

$a_2X_2(\lambda_2 - \lambda_1) = 0 \quad [\because \lambda_1 \neq \lambda_2, X_2 \neq 0]$

$\Rightarrow \quad a_2 = 0$

From equation (1),

$a_1 X_1 = 0$

$\Rightarrow \quad a_1 = 0$

$a_1X_1 + a_2X_2 = 0$

$a_1 = 0$ and $a_2 = 0$

X_1 and X_2 are linearly independent.

Hence, two eigen vectors corresponding to two different eigen values are linearly independent.

Q51. Write any two properties of eigen values and if A and B are non-singular matrices of same order, then show that AB and BA have same eigen values.

Answer :

Properties of Eigen Values

(i) For any square matrix (A), and its transpose (A^T) have same eigen values.

(ii) The sum of eigen values of a matrix is equal to the trace of the matrix.

The given matrices A and B are non-singular matrices.

Consider,

$AB = IAB$

$= (B^{-1}B)$ AB

$\Rightarrow \quad AB = B^{-1} (BA)B$

Consider a matrix A and a non singular matrix K.

The characteristic equation of $K^{-1} AK$ is,

$|K^{-1}AK - \lambda I| = 0$

$\Rightarrow \quad |K^{-1}AK - \lambda K^{-1}IK| = 0 \quad (\because I = K^{-1}IK)$

$\Rightarrow \quad |K^{-1}(A - \lambda I)K| = 0$

$\Rightarrow \quad ||K^{-1} K| \, |A - \lambda I|| = 0 \quad [\because |K^{-1} K| = 1]$

$\Rightarrow \quad |A - \lambda I| = 0$

The characteristic equation of A and $K^{-1} AK$ are same.

$\therefore \quad A$ and $K^{-1}AK$ have same eigen values.

Similarly, $B^{-1}(BA)B$ and BA have same eigen values.

From equation (1), we have,

$B^{-1}(BA)B = AB$

$\therefore \quad AB$ and BA have same eigen value.

Q52. Show that A and A^T has same eigen values but different eigen vectors.

Answer :

Let, $A = [a_{ij}]_{n \times n} \quad [\because i, j = 1, 2, \ldots, n]$

$A^T = [a_{ij}]_{n \times n}$

Consider the characteristic equation of A and A^T.

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & \ldots & a_{2n} \\ . & \ldots & \ldots & \ldots \\ . & \ldots & \ldots & \ldots \\ . & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda \end{vmatrix} = 0 \quad \ldots (1)$$

$$|A^T - \lambda I| = 0 \Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22}-\lambda & \dots & a_{n2} \\ . & \dots & \dots & \dots \\ . & \dots & \dots & \dots \\ . & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn}-\lambda \end{vmatrix} = 0 \quad \dots (2)$$

Since, the interchanging of rows and columns does not alter the determinant.

Equations (1) and (2) are same.

$| A - \lambda I | = | A^T - \lambda I |$

$\therefore$ The characteristic roots of A and A^T are same.

Q53. Prove that distinct characteristic vectors of T corresponding to distinct characteristic values of T are linearly independent.

Answer :

Consider a linear operator T defined on $V(F)$ for which 'm' number of distinct characteristic values are represented as c_1, $c_2,...,c_m$ and their corresponding characteristic vectors are represented as $\alpha_1, \alpha_2,.....,\alpha_m$.

i.e., $T\alpha_i = c_i\, \alpha_i$ for $1 \leq i \leq m$

Let S be a set containing these characteristic vectors

i.e., $S = \{\alpha_1, \alpha_2, ..., \alpha_m\}$

In order to prove that S is linearly independent, induction is performed on m and the number of vectors in S.

For $m = 1$, S contains only one non-zero vector. Hence it is linearly independent.

Consider a linearly independent set

$S_1 = \{\alpha_1,\alpha_2, ..., \alpha_k\}$ for $k < m$

Consider another set S_2 given by,

$S_2 = \{\alpha_1,.....,\alpha_k, \alpha_{k+1}\}$

Let $a_1, ...,a_{k+1} \in F$ such that,

$a_1\, \alpha_1 + ... + a_{k+1}\, \alpha_{k+1} = 0 \quad \dots (1)$

$\Rightarrow \quad T(a_1\, \alpha_1 + ... + a_{k+1}\alpha_{k+1}) = T(0)$

$\Rightarrow \quad a_1\, T(\alpha_1) + ... + a_{k+1}\, T(\alpha_{k+1}) = 0$

$\Rightarrow \quad a_1(c_1\alpha_1) + ... + a_{k+1}\, (c_{k+1}\, \alpha_{k+1}) = 0 \quad \dots (2)$

Multiplying equation (1) with c_{k+1} and subtracting the resultant equation from equation (2),

$a_1(c_1 - c_{k+1})\, \alpha_1 + ... + \alpha_k(c_k - c_{k+1})\, \alpha_k = 0$

Since, c_1,c_{k+1} are all distinct, the vectors $\alpha_1, \alpha_2, ...,\alpha_k$ are linearly independent.

$\therefore \quad a_1 = 0, ..., a_k = 0$

Substituting $a_1 = 0, ...,a_k = 0$ in equation (1),

$a_{k+1}\, \alpha_{k+1} = 0$

Since, $\alpha_{k+1} \neq 0 \;\Rightarrow\; a_{k+1} = 0$

$\therefore \quad a_1 = 0,...., \alpha_k = 0, a_{k+1} = 0$

$\therefore \quad S_2$ is linearly independent.

Hence, it can be concluded that distinct characteristic vectors of T corresponding to distinct characteristic values of T are linearly independent.

Q54. State and prove Cayley-Hamilton theorem.

Answer : Dec.-12, Q16(a)

Statement

For answer refer Unit-1, Q22.

Proof

Let,

$$A = \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1n} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2n} \\ p_{31} & p_{32} & p_{33} & \cdots & p_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{n1} & p_{n2} & p_{n3} & \cdots & p_{nm} \end{bmatrix}$$

The order of matrix A is $n \times n$ whose characteristic equation is,

$$|A - \lambda I| = 0 \quad \text{... (1)}$$

Here, λ – Characteristic root of A and

I – Identity matrix of order $n \times n$

Since equation (1) is a polynomial in λ of degree 'n', it can written as,

$$|A - \lambda I| = (-1)^n [\lambda^n + p_1 \lambda^{n-1} + p_2 \lambda^{n-2} + ... + p_{n-1} \lambda + p_n] = 0 \quad \text{... (2)}$$

Here, $p_1, p_2, p_3 .... p_{n-1}, p_n$ are the constants.

Since the matrix equation $(A - \lambda I)$ has at most 'n' degree in λ, the adjoint matrix i.e., adj $(A - \lambda I)$ will have atmost $(n - 1)$ degree in λ.

$\therefore$ The adj$(A - \lambda I)$ matrix can be written as,

$$\text{Adj}(A - \lambda I) = C_0 \lambda^{n-1} + C_1 \lambda^{n-2} + C_2 \lambda^{n-3} + ... + C_{n-2} \lambda + C_{n-1} \quad \text{... (3)}$$

Here, $C_0, C_1, C_2, ... C_{n-2}, C_{n-1}$ are the matrices of order $n \times n$ and their elements are the functions corresponding to the elements of A.

Where,

$$(A - \lambda I) \cdot \text{Adj } (A - \lambda I) = |A - \lambda I| I \quad \text{... (4)}$$

Substituting equations (2) and (3) in equation (4),

$$(A - \lambda I) [C_0 \lambda^{n-1} + C_1 \lambda^{n-2} + C_2 \lambda^{n-3} + ... + C_{n-2} \lambda + C_{n-1}] = (-1)^n [\lambda^n + p_1 \lambda^{n-1} + p_2 \lambda^{n-2} + ... + p_{n-1} \lambda + p_n] I$$

$$\Rightarrow \quad A C_0 \lambda^{n-1} + A C_1 \lambda^{n-2} + A C_2 \lambda^{n-3} + ... + A C_{n-2} \lambda + AC_{n-1} + (-\lambda I) C_0 \lambda^{n-1} + (-\lambda I) C_1 \lambda^{n-2}$$
$$+ (-\lambda I) C_2 \lambda^{n-3} + ... + (-\lambda I) C_{n-2} \lambda + (-\lambda I) C_{n-1} = (-1)^n \cdot \lambda^n I + (-1)^n p_1 \lambda^{n-1} I + (-1)^n p_2 \lambda^{n-2} I + ...$$
$$+ (-1)^n p_{n-1} \lambda I + (-1)^n p_n I$$

$$\Rightarrow \quad A C_0 \lambda^{n-1} + A C_1 \lambda^{n-2} + A C_2 \lambda^{n-3} + ... + A C_{n-2} \lambda + AC_{n-1} - I C_0 \lambda^{1+n-1} - I C_1 \lambda^{1+n-2} - IC_2 \lambda^{1+n-3} .....$$
$$IC_{n-2} \lambda^{1+1} - IC_{n-1} \lambda = (-1)^n I\lambda^n + (-1)^n p_1 I \lambda^{n-1} + (-1)^n p_2 I \lambda^{n-2} + ... + (-1)^n p_{n-1} I\lambda + (-1)^n p_n I$$

$$\Rightarrow \quad A C_0 \lambda^{n-1} + A C_1 \lambda^{n-2} + A C_2 \lambda^{n-3} + ... + A C_{n-2} \lambda + AC_{n-1} - I C_0 \lambda^n - I C_1 \lambda^{n-1}$$
$$- IC_2 \lambda^{n-2} - ..... - IC_{n-2} \lambda^2 - IC_{n-1} \lambda = (-1)^n I\lambda^n + (-1)^n p_1 I \lambda^{n-1} + (-1)^n I\lambda^{n-2} + ... + (-1) p_{n-1} I\lambda + (-1)^n p_n I$$

$$\Rightarrow \quad - IC_0 \lambda^n + \lambda^{n-1} [AC_0 - IC_1] + \lambda^{n-2} [AC_1 - IC_2] + ... + \lambda [AC_{n-2} + IC_{n-1}] + AC_{n-1} = (-1)^n I \lambda^n + (-1)^n p_1 I\lambda^{n-1}$$
$$+ (-1)^n p_2 I\lambda^{n-2} + ... + (-1)^n p_{n-2} I\lambda + (-1)^n p_n I$$

Comparing the coefficients of $\lambda^n, \lambda^{n-1}, \lambda^{n-2}$ .... on both sides of the above equation,

$$- IC_0 = (-1)^n I \quad \text{... (5)}$$

$$AC_0 - IC_1 = (-1)^n p_1 I \quad \text{... (6)}$$

$$AC_1 - IC_2 = (-1)^n p_2 I \quad \text{... (7)}$$

$$\vdots \qquad \vdots$$

$$AC_{n-2} - IC_{n-1} = (-1)^n p_{n-1} I \quad \text{... (8)}$$

$$AC_{n-1} = (-1)^n p_n I \quad \text{... (9)}$$

Multiplying equations (5), (6), (7), (8) and (9) with $A^n, A^{n-1}, A^{n-2} \ldots A, I$ respectively,

$$\Rightarrow \quad -IC_0 A^n = (-1)^n IA^n \quad \text{... (10)}$$

$$[AC_0 - IC_1]A^{n-1} = (-1)^n p_1 IA^{n-1} \quad \text{... (11)}$$

$$[AC_1 - IC_2]A^{n-2} = (-1)^n p_2 IA^{n-2} \quad \text{... (12)}$$

$$\vdots \qquad \vdots$$

$$[AC_{n-2} - IC_{n-1}]A = (-1)^n p_{n-1} IA \quad \text{... (13)}$$

$$AC_{n-1} I = (-1)^n p_n I.I \quad \text{... (14)}$$

Adding equations (10), (11), (12), (13) and (14),

$$\Rightarrow \quad -IC_0 A^n + [AC_0 - IC_1]A^{n-1} + [AC_1 - IC_2]A^{n-2} + \ldots + [AC_{n-2} - IC_{n-1}]A + [AC_{n-1}].I$$

$$= (-1)^n.IA^n + (-1)^n.p_1 IA^{n-1} + (-1)^n.p_2 IA^{n-2} + \ldots + (-1)^n.p_{n-1} I.A + (-1)^n.p_n I.I$$

$$\Rightarrow \quad -IC_0 A^n + AC_0.A^{n-1} - IC_1 A^{n-1} + AC_1 A^{n-2} - IC_2 A^{n-2} + \ldots + AC_{n-2}.A - IC_{n-1}.A + AC_{n-1} I$$

$$= (-1)^n I[A^n + p_1 A^{n-1} + p_2 A^{n-2} + \ldots + p_{n-1}.A + p_n.I]$$

From L.H.S, considering only C_0 and C_1 terms,

$$\Rightarrow \quad -IC_0 A^n + A^{1+n-1} C_0 - IC_1 A^{n-1} + A^{1+n-2} C_1 - \ldots = (-1)^n I[A^n + p_1 A^{n-1} + p_2 A^{n-2} + \ldots + p_{n-1} \cdot A + p_n I]$$

$$\Rightarrow \quad -IC_0 A^n + A^n C_0 - IC_1 A^{n-1} + A^{n-1} C_1 I + \ldots = (-1)^n I[A^n + p_1 A^{n-1} + p_2 A^{n-2} + \ldots + p_{n-1} A + p_n I]$$

$$\Rightarrow \quad -IC_0 A^n + A^n IC_0 - IC_1 A^{n-1} + A^{n-1} I C_1 + \ldots = (-1)^n I [A^n + p_1 A^{n-1} + p_2 A^{n-2} + \ldots + p_{n-1} A + p_n I]$$

$$[\because \; AI = A \Rightarrow A^n I = A^n]$$

$$I[-C_0 A^n + A^n C_0 - C_1 A^{n-1} + A^{n-1} C_1 + \ldots] = (-1)^n [A^n + p_1 A^{n-1} + p_2 A^{n-2} + \ldots + p_{n-1} A + p_n I]$$

$$\Rightarrow \quad 0 = (-1)^n [A^n + p_1 A^{n-1} + p_2 A^{n-2} + \ldots + p_{n-1} A + p_n I]$$

$$\therefore \quad A^n + p_1 A^{n-1} + p_2 A^{n-2} + \ldots + p_{n-1} A + p_n I = 0$$

Q55. Using Cayley-Hamilton theorem find the inverse of the matrix $A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{pmatrix}$.

Answer : Model Paper-2, Q16(a)

Given matrix is,

$$A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{pmatrix}$$

The characteristic equation of A is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 2 & 3 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & -1 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 2-\lambda & 3 & 4 \\ 0 & 1-\lambda & 5 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (2-\lambda)[(1-\lambda)(-1-\lambda)-0]-3[0]+4[0]=0$

$\Rightarrow \quad (2-\lambda)[-1-\lambda+\lambda+\lambda^2]=0$

$\Rightarrow \quad (2-\lambda)[\lambda^2-1]=0$

$\Rightarrow \quad 2\lambda^2-2-\lambda^3+\lambda=0$

$\Rightarrow \quad -\lambda^3+2\lambda^2+\lambda-2=0$

By Cayley-Hamilton theorem, A should satisfy its characteristic equation such that,

$$-A^3+2A^2+A-2I=0$$

Multiplying above equation with A^{-1},

$$-A^2+2A+I-2A^{-1}=0$$

$$\Rightarrow \quad -A^2+2A+I=2A^{-1}$$

$$\Rightarrow \quad A^{-1}=\frac{1}{2}(-A^2+2A+I) \qquad \ldots (1)$$

$$A^2 = A\,.\,A$$

$$=\begin{bmatrix}2&3&4\\0&1&5\\0&0&-1\end{bmatrix}\begin{bmatrix}2&3&4\\0&1&5\\0&0&-1\end{bmatrix}$$

$$=\begin{bmatrix}4+0+0&6+3+0&8+15-4\\0+0+0&0+1+0&0+5-5\\0+0+0&0+0+0&0+0+1\end{bmatrix}=\begin{bmatrix}4&9&19\\0&1&0\\0&0&1\end{bmatrix}$$

Consider,

$$-A^2+2A+I$$

$$=-\begin{bmatrix}4&9&19\\0&1&0\\0&0&1\end{bmatrix}+2\begin{bmatrix}2&3&4\\0&1&5\\0&0&-1\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}-4&-9&-19\\-0&-1&-0\\-0&-0&-1\end{bmatrix}+\begin{bmatrix}4&6&8\\0&2&10\\0&0&-2\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}-4+4+1&-9+6+0&-19+8+0\\-0+0+0&-1+2+1&-0+10+0\\-0+0+0&-0+0+0&-1-2+1\end{bmatrix}$$

$$=\begin{bmatrix}1&-3&-11\\0&2&10\\0&0&-2\end{bmatrix}$$

Substituting the corresponding values in equation (1),

$$\therefore \quad A^{-1}=\frac{1}{2}\begin{bmatrix}1&-3&-11\\0&2&10\\0&0&-2\end{bmatrix}$$

Q56. Using Cayley-Hamilton theorem find the inverse of the matrix $\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}$.

Answer : Model Paper-3, Q11(b)

Given matrix is,

$$A=\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}$$

The characteristic equation A is $|A-\lambda I|=0$

$$\Rightarrow \left|\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}-\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right|=0$$

$$\Rightarrow \left|\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}-\lambda\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}\right|=0$$

$$\Rightarrow \begin{vmatrix}2-\lambda&-1&1\\-1&2-\lambda&-1\\1&-1&2-\lambda\end{vmatrix}=0$$

$\Rightarrow \quad (2-\lambda)[4-4\lambda+\lambda^2-1]+1[-2+\lambda+1]+1[1-2+\lambda]=0$

$\Rightarrow \quad (2-\lambda)[\lambda^2-4\lambda+3]+\lambda-1+\lambda-1=0$

$\Rightarrow \quad -\lambda^3+4\lambda^2-3\lambda+2\lambda^2-8\lambda+6+\lambda-1+\lambda-1=0$

$\Rightarrow \quad -\lambda^3+6\lambda^2-9\lambda+4=0$

$\Rightarrow \quad \lambda^3-6\lambda^2+9\lambda-4=0$

$\therefore \quad A^3-6A^2+9A-4=0$

Multiply A^{-1} on both sides,

$$A^{-1}(A^3-6A^2+9A-4)=0$$

$$\Rightarrow \quad A^2-6A+9I-4A^{-1}=0$$

$$\Rightarrow \quad A^2-6A+9I=4A^{-1}$$

$$\Rightarrow \quad A^{-1}=\frac{1}{4}(A^2-6A+9I) \qquad \ldots (1)$$

$$A^2=A\times A$$

$$=\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}$$

$$A^2=\begin{bmatrix}6&-5&5\\-5&6&-5\\5&-5&6\end{bmatrix}$$

Substituting the corresponding values in equation (1).

$$A^{-1}=\frac{1}{4}\left[\begin{pmatrix}6&-5&5\\-5&6&-5\\5&-5&6\end{pmatrix}-6\begin{pmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{pmatrix}+9\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right]$$

$$=\frac{1}{4}\left[\begin{pmatrix}6&-5&5\\-5&6&-5\\5&-5&6\end{pmatrix}-\begin{pmatrix}12&-6&6\\-6&12&-6\\6&-6&12\end{pmatrix}+\begin{pmatrix}9&0&0\\0&9&0\\0&0&9\end{pmatrix}\right]$$

$$=\frac{1}{4}\begin{bmatrix}6-12+9&-5+6+0&5-6+0\\-5+6+0&6-12+9&-5+6+0\\5-6+0&-5+6+0&6-12+9\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}3&1&-1\\1&3&1\\-1&1&3\end{bmatrix}$$

$$\therefore \quad A^{-1}=\frac{1}{4}\begin{bmatrix}3&1&-1\\1&3&1\\-1&1&3\end{bmatrix}$$

Q57. If A = $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ then find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + 1$. And also find A^{-1}.

Answer :

The given matrix is,

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

The characteristic equation of matrix A is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(2 - \lambda) ((1 - \lambda) (2 - \lambda)) - 1 (0) + 1 (- (1 - \lambda)) = 0$

$\Rightarrow$ $(2 - \lambda) (1 - \lambda) (2 - \lambda) + (\lambda - 1) = 0$

$\Rightarrow$ $(2 - 2\lambda - \lambda + \lambda^2) (2 - \lambda) + \lambda - 1 = 0$

$\Rightarrow$ $4 - 2\lambda - 4\lambda + 2\lambda^2 - 2\lambda + \lambda^2 + 2\lambda^2 - \lambda^3 + \lambda - 1 = 0$

$\Rightarrow$ $- \lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0$

$\Rightarrow$ $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By Cayley-Hamilton theorem, A should satisfy its characteristic equation i.e.,

$$A^3 - 5A^2 + 7A - 3I = 0 \quad \text{... (1)}$$

Multiplying equation (1) by A^{-1} on both sides,

$A^{-1} (A^3 - 5A^2 + 7A - 3I) = 0$

$\Rightarrow$ $A^2 - 5A + 7I - 3A^{-1} = 0$

$\Rightarrow$ $A^{-1} = \frac{1}{3}(A^2 - 5A + 7I)$

Where,

$$A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$\Rightarrow A^2 - 5A + 7I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

Consider,

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = A^5 (A^3 - 5A^2 + 7A - 3I) + A (A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$$

$$= A^5 (0) + A (0) + A^2 + A + I \qquad [\therefore \text{ From equation (1)}]$$

$$= A^2 + A + I$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

$$\therefore \; A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

Q58. Verify Cayley-Hamilton theorem and find A^5. Where, A = $\begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$.

Answer :

Given matrix is,

$$A = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$,

$$\Rightarrow \quad \left| \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \quad \begin{vmatrix} -\lambda & 2 & -1 \\ -2 & -\lambda & 2 \\ 1 & -2 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow \; -\lambda\,[(-\lambda)(-\lambda) - 2(-2)] - 2\,[(-2)(-\lambda) - (2 \times 1)] - 1[(-2)(-2) - 1(-\lambda)] = 0$

$\Rightarrow \; -\lambda\,[\lambda^2 + 4] - 2\,[2\lambda - 2] - [4 + \lambda] = 0$

$\Rightarrow \; -\lambda^3 - 4\lambda - 4\lambda + 4 - 4 - \lambda = 0$

$\Rightarrow \; -\lambda^3 - 9\lambda = 0$

The matrix A must satisfy the characteristics i.e., by Cayley Hamilton theorem,

$$-A^3 - 9A = 0$$

Where,

$$A^3 = A.A^2$$

$$= \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix} \times \left(\begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix} \times \begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix}$$

$$\therefore \quad A^3 = \begin{bmatrix} 0 & -18 & 9 \\ 18 & 0 & -18 \\ -9 & 18 & 0 \end{bmatrix}$$

$$\therefore \quad -A^3 - 9A = -\begin{bmatrix} 0 & -18 & 9 \\ 18 & 0 & -18 \\ -9 & 18 & 0 \end{bmatrix} - 9\begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 18 & -9 \\ -18 & 0 & 18 \\ 9 & -18 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 18 & -9 \\ -18 & 0 & 18 \\ 9 & -18 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Hence Cayley Hamilton theorem is verified.

Consider,

$$-A^3 - 9A = 0$$

Multiplying with A^2 on both sides,

$$A^5 - 9A^3 = 0 \Rightarrow A^5 = 9A^3$$

$$= 9\begin{bmatrix} 0 & -18 & 9 \\ 18 & 0 & -18 \\ -9 & 18 & 0 \end{bmatrix}$$

$$\therefore \quad A^5 = \begin{bmatrix} 0 & -162 & 81 \\ 162 & 0 & -162 \\ -81 & 162 & 0 \end{bmatrix}$$

1.4 QUADRATIC FORMS, REDUCTION OF QUADRATIC FORM TO CANONICAL FORM BY ORTHOGONAL TRANSFORMATION, NATURE OF QUADRATIC FORMS

Q59. Define quadratic form of a matrix.

Answer :

Quadratic Form

Let A be a symmetric matrix of order n.

Write $A = [a_{ij}]_{n \times n}$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$,

Where $x_1, x_2, ..., x_n$ are n real variables.

Then, $X^T A X = [x_1, x_2, ..., x_n]$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} x_i x_j \qquad ... (1)$$

Where,

$$b_{ij} = \begin{cases} (a_{ij} + a_{ji}) & \text{if } i \neq j \\ a_{ii} & \text{if } i = j \end{cases}$$

$$b_{ij} = \begin{cases} 2a_{ij} & \text{if } i \neq j \\ a_{ii} & \text{if } i = j \end{cases}$$

Since, $a_{ij} = a_{ji},\ \ i \neq j$

The expression on R.H.S. of equation (1) is homogeneous quadratic polynomial in variables $x_1, x_2,..,x_n$ and is called a quadratic form in n variables $x_1, x_2,...,x_2$ corresponding to the symmetric matrix A.

Example

$$\text{Let, } A = \begin{bmatrix} 1 & -2 \\ -2 & 0 \end{bmatrix} \text{ and } X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then the quadratic form corresponding to A is,

$$Q = X^T A X = [x_1\ \ x_2]\begin{bmatrix} 1 & -2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\therefore Q = x_1^2 + 2(-2)\, x_1x_2 + 0.\, x_2^2 = x_1^2 - 4x_1x_2$$

Q60. Write the quadratic form corresponding to the symmetric matrix $\begin{bmatrix} 0 & \frac{5}{2} & 3 \\ \frac{5}{2} & 7 & 1 \\ 3 & 1 & 2 \end{bmatrix}$.

Answer :

Given matrix is,

$$A = \begin{bmatrix} 0 & \frac{5}{2} & 3 \\ \frac{5}{2} & 7 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

The quadratic from corresponding to the symmetric matrix A is given as,

$$X^TAX = [x_1\ \ x_2\ \ x_3]\, A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Substituting the corresponding values in above equation,

$$X^TAX = [x_1\ \ x_2\ \ x_3]\begin{bmatrix} 0 & \frac{5}{2} & 3 \\ \frac{5}{2} & 7 & 1 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \left[0 + \frac{5}{2}x_2 + 3x_3 \quad \frac{5}{2}x_1 + 7x_2 + x_3 \quad 3x_1 + x_2 + 2x_3\right]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \frac{5}{2}x_1 x_2 + 3x_3 x_1 + \frac{5}{2}x_1 x_2 + 7x_2^2 + x_2 x_3 + 3x_1 x_3 + x_2 x_3 + 2x_3^2$$

$$= 5x_1 x_2 + 6x_1 x_3 + 2x_2x_3 + 7x_2^2 + 2x_3^2$$

$$\therefore \quad X^TAX = 7x_2^2 + 2x_3^2 + 5x_1 x_2 + 2x_2 x_3 + 6x_1 x_3 .$$

Q61. Define the following with reference to quadratic form of a matrix,

(a) Positive definite

(b) Negative definite

(c) Positive semi-definite

(d) Negative semi-definite

(e) Indefinite.

Answer :

Consider a matrix 'A' with quadratic form 'X^TAX'.

(a) Positive Definite

A quadratic form X^TAX is said to be positive definite if the eigen values of A are greater than zero.

(or)

If $r = n$ and $s = n$

(b) Negative Definite

A quadratic form X^TAX is said to be negative definite, if the eigen values of A are less than zero.

(or)

If $r = n$ and $s = 0$

(c) Positive Semi-definite

A quadratic form X^TAX is said to be positive semi-definite, if it satisfies the following conditions,

(i) Eigen values of $A \geq 0$

(ii) At least one eigen value should be equal to zero.

(or)

If $r < n$ and $s = r$

(d) Negative Semi-definite

A quadratic form X^TAX is said to be negative semi-definite, if it satisfies the following conditions,

(i) Eigen values of $A \leq 0$

(ii) At least one eigen value should be equal to zero.

(or)

If $r < n$ and $s = 0$

(e) Indefinite

A quadratic form X^TAX is said to be indefinite if it has both positive and negative eigen values.

Q62. Reduce the quadratic form 6x² + 3y² + 3z² – 4xy – 2yz + 4zx to the canonical form also find rank, index and signature of the quadratic form.

Answer : **Model Paper-1, Q11**

Given quadratic form is,

$$6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4zx$$

It can be represented in the matrix form as,

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

The characteristic equation of matrix A is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = 0$$

$\Rightarrow \quad (6-\lambda)[(3-\lambda)(3-\lambda) - (1)] + 2[-2(3-\lambda) + 2] + 2[2 - 2(3-\lambda)] = 0$

$\Rightarrow \quad (6-\lambda)[9 + \lambda^2 - 6\lambda - 1] + 2[-6 + 2\lambda + 2] + 2[2 - 6 + 2\lambda] = 0$

$\Rightarrow \quad (6-\lambda)[\lambda^2 - 6\lambda + 8] + 2[2\lambda - 4] + 2[2\lambda - 4] = 0$

$\Rightarrow \quad 6\lambda^2 - 36\lambda + 48 - \lambda^3 + 6\lambda^2 - 8\lambda + 4\lambda - 8 + 4\lambda - 8 = 0$

$\Rightarrow \quad -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$

$\Rightarrow \quad \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$\Rightarrow \quad \lambda = 8, 2, 2.$

$\therefore$ The eigen values are, $\lambda = 8, 2, 2$

The eigen vectors corresponding to different eigen values are obtained by solving $(A - \lambda I)X = 0$

Where,

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Case (i)

When $\lambda = 8$,

$(A-8I)X_1 = 0$

$$\Rightarrow \left[\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix} - 8\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right]\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}6-8 & -2 & 2\\ -2 & 3-8 & -1\\ 2 & -1 & 3-8\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}-2 & -2 & 2\\ -2 & -5 & -1\\ 2 & -1 & -5\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 + R_1$

$$\Rightarrow \begin{bmatrix}-2 & -2 & 2\\ 0 & -3 & -3\\ 0 & -3 & -3\end{bmatrix}.\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$R_1 \to \dfrac{R_1}{-2}$

$R_2 \to \dfrac{R_2}{-3}$

$R_3 \to \dfrac{R_3}{-3}$

$$\Rightarrow \begin{bmatrix}1 & 1 & -1\\ 0 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\Rightarrow \begin{bmatrix}1 & 1 & -1\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$\Rightarrow \quad x + y - z = 0 \qquad \ldots (1)$

$\Rightarrow \quad y + z = 0 \qquad \ldots (2)$

Substituting $z = k$ in equation (2),

$y + k = 0$

$y = -k$

Substituting $y = -k$ in equation (1),

$x - k - k = 0$

$x = 2k$

$$\therefore\ X_1 = \begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}2k\\ -k\\ k\end{bmatrix} = k\begin{bmatrix}2\\ -1\\ 1\end{bmatrix}$$

Normalizing the above vector, $e_1 = \begin{bmatrix}\frac{2}{\sqrt{6}}\\ \frac{-1}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\end{bmatrix}$

Case (ii)

When $\lambda = 2$

$\Rightarrow \quad (A-2I)X_2 = 0$

$$\Rightarrow \left[\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix} - 2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right]\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}6-2 & -2 & 2\\ -2 & 3-2 & -1\\ 2 & -1 & 3-2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}4 & -2 & 2\\ -2 & 1 & -1\\ 2 & -1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$R_2 \to 2R_2 + R_1$

$R_3 \to 2R_3 - R_1$

$$\Rightarrow \begin{bmatrix}4 & -2 & 2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$\Rightarrow \quad 4x - 2y + 2z = 0$

$\Rightarrow \quad 2x - y + z = 0$

Let, $x = k_1$, $y = k_2$,

$z = -2x + y$

$= -2k_1 + k_2$

$$= \begin{bmatrix}k_1 + 0k_2\\ 0k_1 + 1k_2\\ -2k + k_2\end{bmatrix} = k_1\begin{bmatrix}1\\ 0\\ -2\end{bmatrix} + k_2\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$$

$$X_2 = \begin{bmatrix}1\\ 0\\ -2\end{bmatrix}$$

Normalizing the above vector,

$$e_2 = \begin{bmatrix}\frac{1}{\sqrt{5}}\\ 0\\ \frac{-2}{\sqrt{5}}\end{bmatrix}$$

$$X_3 = \begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$$

Normalizing the above vector,

$$e_3 = \begin{bmatrix}0\\ \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}$$

The Eigen vectors of symmetric matrix are orthogonal.

$\therefore \quad e_1(e_2)^T = 0, e_1(e_3)^T = 0$, but $e_2.e_3^T = \dfrac{-2}{\sqrt{10}} \neq 0$.

Hence, the vector X_2 is,

$X_1X_2^T = 0,\ X_1X_3^T = 0,\ X_2X_3^T = 0$

Let, $X_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be the required Eigen vector,

$$X_1X_2^T = 0 \Rightarrow \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}[a, b, c] = 0$$

$$2a - b + c = 0 \qquad \text{... (3)}$$

$$X_2X_3^T = 0 \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix}[0, 1, 1] = 0$$

$$b + c = 0 \qquad \text{... (4)}$$

Solving equations (3) and (4),

$$\therefore \quad X_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$\therefore$ The normalized vector is, $\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{bmatrix}$

$\therefore$ The normalised matrix is, $N = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

The diagonal matrix is, $D = N^TAN$

$$\begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The canonical form is,

$$[y_1\ y_2\ y_3]\, D \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = [y_1\ y_2\ y_3] \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 8y_1^2 + 2y_2^2 + 2y_3^2$$

Rank of matrix = Number of non-zero rows.

$\therefore \quad r = 3$

Index = Number of positive terms

$s = 3$

Signature = $2s - r = 2(3) - 3 = 3$

$\therefore$ Signature = 3.

Q63. Reduce the quadratic form $2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx$ to canonical form by orthogonal transformation and hence find rank, index, signature and nature of the quadratic form.

Answer :

Given quadratic from is,

$$2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx$$

The above expression in the matrix form is written as,

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

The characteristic equation of A is given as $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left[(2-\lambda)^2 - (-1)(-1)\right] - (-1)\left[-1(2-\lambda) - (-1)(-1)\right] - 1\left[(-1)(-1) - (-1)(2-\lambda)\right] = 0$$

$$\Rightarrow (2-\lambda)(4+\lambda^2 - 4\lambda - 1) + (-2+\lambda-1) - (1+2-\lambda) = 0$$

$$\Rightarrow 8 + 2\lambda^2 - 8\lambda - 2 - 4\lambda - \lambda^3 + 4\lambda^2 + \lambda - 2 + \lambda - 1 - 1 - 2 + \lambda = 0$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 - 9\lambda = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda = 0$$

$$\Rightarrow \lambda(\lambda^2 - 6\lambda + 9) = 0$$

$$\Rightarrow \lambda = 0, \lambda^2 - 6\lambda + 9 = 0$$

$$\Rightarrow \lambda = 0, \ \lambda^2 - 3\lambda - 3\lambda + 9 = 0$$

$$\Rightarrow \lambda = 0, \ \lambda(\lambda - 3) - 3(\lambda - 3) = 0$$

$$\Rightarrow \lambda = 0, \ (\lambda - 3)(\lambda - 3) = 0$$

$$\Rightarrow \lambda = 0, \lambda = 3, 3$$

$\therefore$ Eigen values are, $\lambda = 0, 3, 3$.

The eigen vector corresponding to different eigen values are obtained by solving $(A - \lambda I)X = 0$

Where,

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Case (i)

When $\lambda = 0$,

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \to 2R_2 + R_1$, $R_3 \to 2R_3 + R_1$

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 + R_2$

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 - x_2 - x_3 = 0;$

$3x_2 - 3x_3 = 0 \Rightarrow x_2 = x_3$

Let $x_2 = x_3 = k$

Then, $2x_1 - k - k = 0$

$\Rightarrow \quad 2x_1 - 2k = 0$

$\Rightarrow \quad x_1 = k$

$$\Rightarrow \quad X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore \quad X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Case (ii)

When $\lambda = 3$,

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 - x_2 - x_3 = 0$

$\Rightarrow \quad x_1 + x_2 + x_3 = 0$

Let, $x_2 = 0$;

Then $x_1 + 0 + x_3 = 0$

$\Rightarrow \quad x_3 = -\ x_1$

Let, $x_3 = k$, then $x_1 = -k$

$$\Rightarrow \quad X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} = k\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \quad X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence, X_3 can be determined using the orthogonal property,

i.e., $X_1^T . X_3 = 0$ and $X_2^T . X_3 = 0$

$$\Rightarrow \quad [1\ 1\ 1]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \ ; \ [-1\ 0\ 1]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow \quad x_1 + x_2 + x_3 = 0 \ ; -x_1 + x_3 = 0$

Solving the above two equations,

$x_2 + 2x_3 = 0$

Let $x_3 = k$; $x_2 + 2k = 0$

$x_2 = -2k$

Then, $x_1 - 2k + k = 0$

$\Rightarrow \quad x_1 - k = 0$

$\Rightarrow \quad x_1 = k$

$$\Rightarrow \quad X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ -2k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\therefore \quad X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The normalised matrix (N) is given as,

$$N = \begin{bmatrix} \frac{X_1}{|X_1|} & \frac{X_2}{|X_2|} & \frac{X_3}{|X_3|} \end{bmatrix}$$

$$\Rightarrow \quad N = \begin{bmatrix} \frac{1}{\sqrt{1+1+1}} & \frac{-1}{\sqrt{1+0+1}} & \frac{1}{\sqrt{1+4+1}} \\ \frac{1}{\sqrt{1+1+1}} & 0 & \frac{-2}{\sqrt{1+4+1}} \\ \frac{1}{\sqrt{1+1+1}} & \frac{1}{\sqrt{1+0+1}} & \frac{1}{\sqrt{1+4+1}} \end{bmatrix}$$

$$\Rightarrow \quad N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\Rightarrow \quad N^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Diagonal matrix $D = N^TAN$

$$\Rightarrow D = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\Rightarrow \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Canonical form is given as,

$C.F = Y^T DY$

$$\Rightarrow \quad C.F = [y_1 \ y_2 \ y_3] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= [0 \ \ 3y_2 \ \ 3y_3] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 0y_1^2 + 3y_2^2 + 3y_3^2$$

$$= 3y_2^2 + 3y_3^2$$

$\therefore$ Canonical form $(C.F) = 3y_2^2 + 3y_3^2$

Rank of the quadratic form = 2

Index = 2

Signature = 2

It is positive semi definite.

Q64. Reduce the quadratic form 3x² + 5y² + 3z² – 2xy – 2yz + 2zx to canonical form by orthogonal transformation and hence find the rank, index, signature and nature of the quadratic form.

Answer : Model Paper-2, Q11

Given quadratic form is,

$3x^2 + 5y^2 + 3z^2 - 2xy - 2yz + 2zx$

The above expression in the matrix form is written as,

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

The characteristic equation of A is given as,

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (3-\lambda)\,[(5-\lambda)\,(3-\lambda) - (-1)(-1)] - (-1)\,[-1(3-\lambda) - 1(-1)] + 1\,[(-1)(-1) - 1(5-\lambda)] = 0$

$\Rightarrow \quad (3-\lambda)\,[15 - 5\lambda - 3\lambda + \lambda^2 - 1] + 1[-3 + \ + 1] + 1[1 - 5 + \lambda] = 0$

$\Rightarrow \quad (3-\lambda)[\lambda^2 - 8\lambda + 14] - 3 + \lambda + 1 + 1 - 5 + \lambda = 0$

$\Rightarrow \quad 3\lambda^2 - 24\lambda + 42 - \lambda^3 + 8\lambda^2 - 14\lambda - 6 + 2\lambda = 0$

$\Rightarrow \quad -\lambda^3 + 11\lambda^2 - 36\lambda + 36 = 0$

$\Rightarrow \quad \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

By trial and error method,

λ = 2				
	1	–11	36	–36
	0	2	–18	36
	1	–9	18	0

$\lambda = 2, \ \lambda^2 - 9\lambda + 18 = 0$

$\lambda = 2, \ \lambda^2 - 6\lambda - 3\lambda + 18 = 0$

$\lambda = 2, \ \lambda(\lambda - 6) - 3(\lambda - 6) = 0$

$\lambda = 2, \ (\lambda - 3)\,(\lambda - 6) = 0$

$\lambda = 2, \ \ \lambda = 3, \lambda = 6$

$\therefore$ Eigen values are $\lambda = 2, 3, 6$

The eigen vectors corresponding to different eigen values obtained by solving $(A - \lambda I)X = 0$

Where, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Case (i)

When $\lambda = 2$,

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \to R_2 + R_1$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_2 = 0 \Rightarrow x_2 = 0$

$x_1 - x_2 + x_3 = 0 \Rightarrow x_1 - 0 + x_3 = 0$

$\Rightarrow \quad x_1 = -x_3$

Let $x_3 = k$ then $x_1 = -k$

$$\therefore \quad X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k \\ 0 \\ k \end{bmatrix} = k\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \quad X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Case (ii)

When $\lambda = 3$,

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2 + R_1$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_2 + x_3 = 0$

$\Rightarrow \quad x_2 = x_3$

Let, $x_2 = k$

$-x_1 + 2x_2 - x_3 = 0 \Rightarrow -x_1 + 2k - k = 0$

$\Rightarrow -x_1 + k = 0$

$\Rightarrow \quad x_1 = k$

$$\Rightarrow \quad X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case (iii)

When $\lambda = 6$,

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_2 \to 3R_2 - R_1$, $3R_3 + R_1$

$$\begin{bmatrix} -3 & -1 & 1 \\ 0 & -2 & -4 \\ 0 & -4 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} -3 & -1 & 1 \\ 0 & -2 & -4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x_2 - 4x_3 = 0$

$\Rightarrow \quad x_2 = -2x_3$

Let $x_3 = k$ then $x_2 = -2k$

$-3x_1 - k_2 + k_3 = 0$

$\Rightarrow \quad -3x_1 + 2k + k = 0$

$\Rightarrow \quad -3x + 3k = 0$

$\Rightarrow \quad x_1 = k$

$$\Rightarrow \quad X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ -2k \\ k \end{bmatrix} = k\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

The normalised matrix (N) is given as,

$$N = \begin{bmatrix} \frac{X_1}{|X_1|} & \frac{X_2}{|X_2|} & \frac{X_3}{|X_3|} \end{bmatrix}$$

$$\Rightarrow \quad N = \begin{bmatrix} \frac{-1}{\sqrt{1+0+1}} & \frac{1}{\sqrt{1+1+1}} & \frac{1}{1+4+1} \\ 0 & \frac{1}{\sqrt{1+1+1}} & \frac{-2}{\sqrt{1+4+1}} \\ \frac{1}{\sqrt{1+0+1}} & \frac{1}{\sqrt{1+1+1}} & \frac{1}{\sqrt{1+4+1}} \end{bmatrix}$$

$$\Rightarrow \quad N = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\Rightarrow \quad N^T = \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

Diagonal matrix $(D) = N^T AN$

$$\Rightarrow \quad D = \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\Rightarrow \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$\therefore$ Canonical form is given as,

$C.F = Y^T DY$

$$= [y_1 \ \ y_2 \ \ y_3] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= [2y_1 \ \ 3y_2 \ \ 6y_3] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 2y_1^2 + 3y_2^2 + 6y_3^2$$

Rank of the quadratic form = 3

Index = 3

Signature = 3

It is positive definite.

Q65. Reduce the quadratic form $2x^2 + 5y^2 + 3z^2 + 4xy$ to a canonical form through an orthogonal transformation. Find also its nature.

Answer :

Given quadratic form is,

$$2x^2 + 5y^2 + 3z^2 + 4xy \quad \text{... (1)}$$

Equation (1) can be represented in matrix form as,

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The characteristic equation of matrix A is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 2-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (2 - \lambda)((5 - \lambda)(3 - \lambda) - 0) - 2(2(3 - \lambda)) - 0 + 0 = 0$

$\Rightarrow \quad (2 - \lambda)(15 - 5\lambda - 3\lambda + \lambda^2) - 2(6 - 2\lambda) = 0$

$\Rightarrow \quad (2 - \lambda)(\lambda^2 - 8\lambda + 15) - 12 + 4\lambda = 0$

$\Rightarrow \quad 2\lambda^2 - 16\lambda + 30 - \lambda^3 + 8\lambda^2 - 15\lambda - 12 + 4\lambda = 0$

$\Rightarrow \quad -\lambda^3 + 10\lambda^2 - 27\lambda + 18 = 0$

$\Rightarrow \quad \lambda^3 - 10\lambda^2 + 27\lambda - 18 = 0$

$\Rightarrow \quad \lambda = 6, \lambda = 3, \lambda = 1$

$\therefore$ The eigen values obtained are $\lambda = 6, \lambda = 3, \lambda = 1$.

Case (i): When $\lambda = 6$

Characteristic equation is,

$(A - \lambda I)X = 0$

$\Rightarrow \quad (A - 6I)X = 0$

$$\Rightarrow \quad \left[\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2-6 & 2 & 0 \\ 2 & 5-6 & 0 \\ 0 & 0 & 3-6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} -4 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 2R_2 + R_1$

$$\Rightarrow \quad \begin{bmatrix} -4 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-4x + 2y = 0$

$-3z = 0$

Let, $y = k_1$ and $z = 0$

$\Rightarrow \quad -4x = -2k_1$

$x = \frac{1}{2}k_1$

$\therefore$ The eigen vectors corresponding to $\lambda = 6$ are,

$$X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{k_1}{2} \\ k_1 \\ 0 \end{bmatrix} = k_1 \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

Case (ii) When $\lambda = 3$

Characteristic equation is,

$(A - \lambda I)X = 0$

$\Rightarrow \quad (A - 3I)X = 0$

$$\Rightarrow \quad \left[\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2-3 & 2 & 0 \\ 2 & 5-3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -x + 2y = 0$ ... (1)

$2x + 2y = 0$ ... (2)

Solving equations (1) and (2)

	x		y		z	
2		0		–1		2
2		0		2		2

$$\Rightarrow \frac{x}{0} = \frac{y}{0} = \frac{z}{-2-4}$$

$$\Rightarrow X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -6 \end{bmatrix}$$

Case (iii) When $\lambda = 1$

Characteristic equation is,

$(A - \lambda I)X = 0$

$\Rightarrow \quad (A - I)X = 0$

$$\Rightarrow \left[\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2-1 & 2 & 0 \\ 2 & 5-1 & 0 \\ 0 & 0 & 3-1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x + 2y = 0$

$2z = 0$

Let, $y = k_1$

$\Rightarrow \quad x + 2k_1 = 0$

$x = -2k_1$

$2z = 0 \Rightarrow z = 0$

$$\Rightarrow X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2k_1 \\ k_1 \\ 0 \end{bmatrix} = k_1\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore X_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} 0 \\ 0 \\ -6 \end{bmatrix}, X_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Hence X_1, X_2, X_3 are pair wise orthogonal.

Normalising the vectors,

$$\hat{e}_1 = \frac{X_1}{|X_1|} = \begin{bmatrix} \dfrac{\frac{1}{2}}{\sqrt{\left(\frac{1}{2}\right)^2 + (1)^2 + (0)^2}} \\ \dfrac{1}{\sqrt{\left(\frac{1}{2}\right)^2 + (1)^2 + (0)^2}} \\ \dfrac{0}{\sqrt{\left(\frac{1}{2}\right)^2 + (1)^2 + (0)^2}} \end{bmatrix} = \begin{bmatrix} \dfrac{\frac{1}{2}}{\sqrt{\frac{5}{4}}} \\ \dfrac{1}{\sqrt{\frac{5}{4}}} \\ 0 \end{bmatrix}$$

$$\hat{e}_2 = \begin{bmatrix} \dfrac{0}{\sqrt{0^2 + 0^2 + (-6)^2}} \\ \dfrac{1}{\sqrt{0^2 + 0^2 + (-6)^2}} \\ \dfrac{-6}{\sqrt{0^2 + 0^2 + (-6)^2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\hat{e}_3 = \begin{bmatrix} \dfrac{-2}{\sqrt{(-2)^2 + (1)^2 + (0)^2}} \\ \dfrac{1}{\sqrt{(-2)^2 + (1)^2 + (0)^2}} \\ \dfrac{0}{\sqrt{(-2)^2 + (1)^2 + (0)^2}} \end{bmatrix} = \begin{bmatrix} -\dfrac{2}{\sqrt{5}} \\ \dfrac{1}{\sqrt{5}} \\ 0 \end{bmatrix}$$

The normalised matrix, 'N' is,

$$N = [\hat{e}_1 \quad \hat{e}_2 \quad \hat{e}_3]$$

$$N = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & -1 & 0 \end{bmatrix}$$

The diagonal matrix D is,

$$D = N^T A N$$

$$\Rightarrow \quad D = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & -1 & 0 \end{bmatrix}^T \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & -1 & 0 \end{bmatrix}$$

$$\Rightarrow \quad D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ The canonical form is given as,

$$Y^T D Y = [y_1 \quad y_2 \quad y_3] \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= [6y_1 \quad 3y_2 \quad y_3] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 6y_1^2 + 3y_2^2 + y_3^2$$

$\therefore$ The canonical form is,

$$6y_1^2 + 3y_2^2 + y_3^2$$

Here, all the eigen values are positive.

$\therefore$ Nature of the quadratic form is positive definite.

Q66. Reduce the quadratic form 2xy + 2xz + 2yz to a canonical form and also find its nature of the matrix.

Answer : May/June-12, Q11(b)

The given quadratic form is,

$$2xy + 2xz + 2yz$$

Then, the quadratic form can be represented in matrix form as,

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

The characteristic equation of the matrix A is,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda(\lambda^2 - 1) - 1(-\lambda - 1) + 1(1 + \lambda) = 0$$

$$\Rightarrow -\lambda^3 + \lambda + \lambda + 1 + 1 + \lambda = 0$$

$$\Rightarrow -\lambda^3 + 3\lambda + 2 = 0$$

$$\Rightarrow \lambda^3 - 3\lambda - 2 = 0$$

$$\begin{array}{r|rrrr} = -1 & 1 & 0 & -3 & -2 \\ & 0 & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & \underline{|0} \end{array}$$

$$\Rightarrow (\lambda + 1)(\lambda^2 - \lambda - 2) = 0$$

$$\Rightarrow (\lambda + 1)(\lambda^2 - 2\lambda + \lambda - 2) = 0$$

$$\Rightarrow (\lambda + 1)(\lambda(\lambda - 2) + 1(\lambda - 2)) = 0$$

$$\Rightarrow (\lambda + 1)(\lambda - 2)(\lambda + 1) = 0$$

$$\Rightarrow \lambda = -1, 2, -1$$

$\therefore$ The eigen values are, $\lambda = -1, 2, -1$

The eigen vector corresponding to different eigen values are obtained by solving $(A - \lambda I)X = 0$

Where,

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Case (i): $\lambda = -1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 + x_2 + x_3 = 0$$

Let, $x_2 = 0$

$$\Rightarrow x_1 = -x_3$$

$\therefore$ The eigen vector is,

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Case (ii): $\lambda = 3$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 + x_2 + x_3 = 0 \qquad \text{... (1)}$$

$$x_1 - 2x_2 + x_3 = 0 \qquad \text{... (2)}$$

Solving equations (1) and (2),

$$\frac{x_1}{1+2} = \frac{-x_1}{-2-1} = \frac{x_3}{4-1}$$

$$\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3}$$

$\therefore$ The eigen vector is,

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The third eigen vector (X_3) can be determined as,

$$\text{Let, } X_3 = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Since X_3 is orthogonal to X_1 and X_2, we get two equations corresponding to eigen vectors X_1 and X_2 i.e.,

$$p + 0.q - r = 0 \quad \text{... (3)}$$

$$p + q + r = 0 \quad \text{... (4)}$$

Solving equations (3) and (4),

$$\frac{p}{0+1} = \frac{-q}{1+1} = \frac{r}{1-0}$$

$$\Rightarrow \quad \frac{p}{1} = \frac{q}{-2} = \frac{r}{1}$$

$\therefore$ The eigen vector is,

$$X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\therefore \quad X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Normalizing the above three eigen vector,

$$e_1 = \begin{bmatrix} \frac{1}{\sqrt{1^2+0^2+(-1)^2}} \\ \frac{0}{\sqrt{1^2+0^2+(-1)^2}} \\ \frac{-1}{\sqrt{1^2+0^2+(-1)^2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

$$e_2 = \begin{bmatrix} \frac{1}{\sqrt{1^2+1^2+1^2}} \\ \frac{1}{\sqrt{1^2+1^2+1^2}} \\ \frac{1}{\sqrt{1^2+1^2+1^2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$e_3 = \begin{bmatrix} \frac{1}{\sqrt{1^2+(-2)^2+1^2}} \\ \frac{-2}{\sqrt{1^2+(-2)^2+1^2}} \\ \frac{1}{\sqrt{1^2+(-2)^2+1^2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

The normalized matrix, 'N' is,

$$N = [e_1 e_2\ e_3]$$

$$N = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$\Rightarrow \quad N = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

The diagonal matrix D is,

$$D = N^T AN$$

$$D = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{2}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow \quad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\therefore$ The canonical form is given as,

$$Y^T DY = [y_1\ y_2\ y_3] \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= [-y_1 \quad 2y_2 \quad -y_3] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= -y_1^2 + 2\, y_2^2 - y_3^2$$

$\therefore$ The canonical form is,

$$-y_1^2 + 2\, y_2^2 - y_3^2$$

Since some of the eigen values are positive and some are negative, therefore the nature of quadratic form is indefinite.

Q67. Reduce the quadratic form $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2 x_3$ by orthogonal transformation. Find rank signature and nature.

Answer : **Jan.-12, Q11(b)**

The symmetric matrix A corresponding to the given quadratic form Q is,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

The characteristic equation is $|A - \lambda I| = 0$.

$$\left| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

Applying $C_2 \to C_2 + C_3$

$$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & -1 \\ 0 & 2-\lambda & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)(2-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 3-\lambda \end{vmatrix} = 0$$

Applying $R_3 \to R_3 - R_2$

$$\Rightarrow \quad (1-\lambda)(2-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 4-\lambda \end{vmatrix} = 0$$

Applying $C_3 \to C_3 + C_2$

$$\Rightarrow \quad (1-\lambda)(2-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (1-\lambda)(2-\lambda)(4-\lambda) = 0$

$\therefore \quad \lambda = 1, 2$ and 4 are the eigen values of A.

The eigen vectors corresponding to different eigen values are obtained by solving $(A - \lambda I)X = 0$.

Where,

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Case (i)

When $\lambda = 1$

$\Rightarrow \quad (A - I)X = 0$

$$\Rightarrow \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_3 \to 2R_3 + R_2$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 2x_2 - x_3 = 0$ and $3x_3 = 0$

$\Rightarrow \quad x_2 = 0$ and $x_3 = 0$

Let, $x_1 = k_1$, arbitrary. Then, the eigen vectors corresponding to $\lambda = 1$ are,

$$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Case (ii)

When $\lambda = 2$

$\Rightarrow \quad (A - 2I)X = 0$

$$\Rightarrow \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Applying $R_3 \to R_3 + R_2$

$$\Rightarrow \quad \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -x_1 = 0$ and $x_2 - x_3 = 0$

$\Rightarrow \quad x_1 = 0$ and $x_2 = x_3$

Let, $x_2 = k_2$ be arbitrary so that $x_3 = k_2$

Hence, all the eigen vectors corresponding to $\lambda = 2$ are,

$$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ k_2 \\ k_2 \end{bmatrix} = k_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Case (iii)

When $\lambda = 4$

$\Rightarrow \quad (A - 4I)X = 0$

$$\Rightarrow \quad \begin{bmatrix} -3 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \to -\frac{1}{3}R_1$

$R_3 \to R_3 - R_2$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_1 = 0$ and $-x_2 - x_3 = 0$

$\Rightarrow \quad x_1 = 0$ and $x_2 = -x_3$

Let $x_3 = k_3$ be arbitrary. Then, $x_2 = -k_3$

Hence, all the eigen vectors corresponding to $\lambda = 4$ are,

$$X_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -k_3 \\ k_3 \end{bmatrix} = k_3\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\therefore \quad X_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Thus, $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ are eigen vectors corresponding to distinct eigen values (λ = 1, 2 and 4) of the symmetric matrix A.

Hence X_1, X_2 and X_3 are pair wise orthogonal.

The inner products,

$$X_1^T X_2 = X_2^T X_3 = X_3^T X_1 = 0$$

Normalizing the eigen vectors,

$$e_1 = \frac{X_1}{|X_1|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$e_2 = \frac{X_2}{|X_2|} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$e_3 = \frac{X_3}{|X_3|} = \begin{bmatrix} 0 \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Hence, $N = [e_1 \quad e_2 \quad e_3]$ is the normalised matrix,

$$\Rightarrow \quad N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

The diagonal matrix is,

$$D = N^T AN$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{2}} & \frac{2}{\sqrt{2}} \\ 0 & -\frac{4}{\sqrt{2}} & \frac{4}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$\therefore$ The canonical form is,

$$Y^T DY = [y_1 \quad y_2 \quad y_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= y_1^2 + 2y_2^2 + 4y_3^2$$

No. of positive terms (index) (s) = 3

rank of quadratic form, (r) = 3

signature of quadratic form = $2s - r$

The given quadratic form is positive definite.

UNIT 2

DIFFERENTIAL EQUATIONS OF FIRST ORDER

PART-A

SHORT QUESTIONS WITH SOLUTIONS

Q1. What is exact differential equation?

Answer :

Exact Differential Equation

An equation of the form $Mdx + Ndy = 0$ (where M and N are functions of x and y) is said to be Exact differential equation.

i.e., $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Example

$(x^2 + y)dx + (y^2 + x)dy = 0$

The general solution of this form is obtained as,

$$\underset{\text{(y constnat)}}{\int} Mdx + \underset{\text{(terms independent of } x)}{\int} Ndy = c$$

Q2. Solve $(3x^2 + 2e^y)dx + (2xe^y + 3y^2)dy = 0$.

Answer : **(Model Paper-1, Q3 | June-11, Q2)**

Given differential equation is,

$$(3x^2 + 2e^y)dx + (2xe^y + 3y^2)dy = 0 \quad \text{... (1)}$$

Equation (1) is of the form $Mdx + Ndy = 0$ ... (2)

Comparing equations (1) and (2),

$M = 3x^2 + 2e^y$ and $N = 2xe^y + 3y^2$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[3x^2 + 2e^y]$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[3x^2] + \frac{\partial}{\partial y}[2e^y]$$

$$\Rightarrow \frac{\partial M}{\partial y} = 0 + 2\frac{\partial}{\partial y}(e^y)$$

$$\therefore \frac{\partial M}{\partial y} = 2e^y \quad \text{... (3)}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[2xe^y + 3y^2]$$

$$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}[2xe^y] + \frac{\partial}{\partial x}[3y^2]$$

$$\Rightarrow \frac{\partial N}{\partial x} = 2e^y \frac{\partial}{\partial x}(x) + 0$$

$$\Rightarrow \frac{\partial N}{\partial x} = 2e^y(1)$$

$$\therefore \frac{\partial N}{\partial x} = 2e^y \quad \text{... (4)}$$

From equations (3) and (4),

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Equation (1) is an exact differential equation.

$\therefore$ The general solution is,

$$\int_{(y \text{ constant})} Mdx + \int_{(\text{terms independent of } x)} Ndy = c \quad \text{... (5)}$$

Substituting the corresponding values in equation (5),

$$\Rightarrow \int (3x^2 + 2e^y)dx + \int 3y^2 dy = c$$

$$\Rightarrow \int 3x^2 dx + \int 2e^y dx + \int 3y^2 dy = c$$

$$\Rightarrow 3\int x^2 .dx + 2e^y \int dx + 3\int y^2\, dy = c$$

$$\Rightarrow 3\left(\frac{x^3}{3}\right) + 2e^y(x) + 3\left(\frac{y^3}{3}\right) = c$$

$$\Rightarrow x^3 + 2xe^y + y^3 = c$$

$\therefore$ The general solution is $x^3 + y^3 + 2xe^y = c$

Q3. Find the solution of the differential equation, (y – x + 1)dy – (y + x + 2)dx = 0

Answer : Jan.-12, Q2

Given differential equation is,

$$(y - x + 1)dy - (y + x + 2)dx = 0$$

$$\Rightarrow -(y + x + 2)dx + (y - x + 1)dy = 0 \quad \text{... (1)}$$

Equation (1) is of the form $Mdx + Ndy = 0$. ... (2)

Comparing equations (1) and (2)

$M = -(y + x + 2)$ and $N = (y - x + 1)$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}[-(y + x + 2)]$$

$$\Rightarrow \frac{\partial M}{\partial y} = -\left[\frac{\partial}{\partial y}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial y}(2)\right]$$

$$\Rightarrow \frac{\partial M}{\partial y} = -[1 + 0 + 0]$$

$$\therefore \frac{\partial M}{\partial y} = -1 \quad \text{... (3)}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(y - x + 1)$$

$$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial x}(1)$$

$$\Rightarrow \frac{\partial N}{\partial x} = 0 - 1 + 0$$

$$\Rightarrow \frac{\partial N}{\partial x} = -1 \quad \text{... (4)}$$

From equations (3) and (4),

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ Equation (1) is an exact differential equation.

$\therefore$ The general solution of equation (1) is given as,

$$\int_{(y \text{ constant})} Mdx + \int_{(\text{terms independent of } x)} Ndy = c \quad \text{... (5)}$$

Substituting the corresponding values in equation (5),

$$\Rightarrow \int -(y + x + 2)dx + \int (y + 1)dy = c$$

$$\Rightarrow -\left[\int ydx + \int xdx + \int 2dx\right] + \left[\int ydy + \int dy\right]$$

$$\Rightarrow -\left[y(x) + \frac{x^2}{2} + 2(x)\right] + \left[\frac{y^2}{2} + y\right] = c$$

$$\Rightarrow -\left[\frac{2xy + x^2 + 4x}{2}\right] + \frac{y^2 + 2y}{2} = c$$

$$\Rightarrow \frac{-(2xy + x^2 + 4x) + y^2 + 2y}{2} = c$$

$$\Rightarrow -2xy - x^2 - 4x + y^2 + 2y = 2c$$

$$\Rightarrow y^2 - x^2 - 2xy + 2y - 4x = 2c$$

$\therefore$ The required solution is, $y^2 - x^2 - 2xy + 2y - 4x = 2c$

Q4. Solve $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$.

Answer : (June-13, Q2 | May/June-12, Q1)

Given differential equation is,

$$\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(y\cos x + \sin y + y)}{(\sin x + x\cos y + x)}$$

$$\Rightarrow (\sin x + x\cos y + x)dy = -(y\cos x + \sin y + y)dx$$

$$\Rightarrow (\sin x + x\cos y + x)\, dy + (y\cos x + \sin y + y)dx = 0$$

$$\Rightarrow (y\cos x + \sin y + y)dx + (\sin x + x\cos y + x)\, dy = 0 \quad \text{... (1)}$$

Equation (1) of the form is, $Mdx + Ndy = 0$ ... (2)

Comparing equations (1) and (2),

$M = y\cos x + \sin y + y$ and $N = \sin x + x\cos y + x$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y\cos x + \sin y + y)$$

$$\Rightarrow \frac{\partial M}{\partial y} = \cos x + \cos y + 1 \quad \text{... (3)}$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(\sin x + x\cos y + x)$$

$$\Rightarrow \frac{\partial N}{\partial x} = \cos x + \cos y + 1 \quad \text{... (4)}$$

From equations (3) and (4),

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Equation (1) is an exact differential equation,

The general solution is given as,

$$\underset{(y \text{ constant})}{\int} Mdx + \underset{(\text{terms independent of } x)}{\int} Ndy = c \quad \text{... (5)}$$

Substituting the corresponding values in equation (5),

$$\int (y\cos x + \sin y + y)dx + \int 0dy = c$$

$$\Rightarrow y(\sin x) + x\sin y + xy = c$$

$\therefore$ $x\sin y + y\sin x + xy = c$ is the required general solution.

Q5. Define an integrating factor.

Answer :

A function $F(x, y)$ which can make a non-exact differential equation of the type $M(x, y)\,dx + N(x, y)\,dy = 0$ exact, is called the integrating factor of the differential equation.

Q6. I.F of $xy(1 + xy^2)\frac{dy}{dx} = 1$ is,

Answer : **(Model Paper-2, Q3 | May/June-12, Q2)**

Given differential equation is,

$$xy(1 + xy^2)\frac{dy}{dx} = 1$$

$$\Rightarrow (1 + xy^2)\,dy = \frac{1}{xy}dx$$

$$\Rightarrow \frac{1}{xy}dx - (1 + xy^2)\,dy = 0 \quad \text{... (1)}$$

Equation (1) is of the form,

$$Mdx + Ndy = 0 \quad \text{... (2)}$$

Comparing equations (1) and (2),

$$M = \frac{1}{xy} \text{ and } N = -(1 + xy^2)$$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{-1}{xy^2}, \frac{\partial N}{\partial x} = -y^2$$

Since, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ the integrating factor is given by,

$$\text{I.F} = \frac{1}{Mx + Ny} \quad \text{... (3)}$$

Substituting the corresponding values in equation (3),

$$\text{I.F} = \frac{1}{\left(\frac{1}{xy}\right)x - (1 + xy^2)y}$$

$$= \frac{1}{\frac{1}{y} - y - xy^3} = \frac{y}{1 - y^2(1 + xy^2)}$$

$$\therefore \text{I.F} = \frac{y}{1 - y^2(1 + xy^2)}$$

Q7. Find an integrating factor of $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$.

Answer : **(Model Paper-3, Q3 | Dec.-13, Q1)**

Given differential equation is,

$$(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0 \quad \text{... (1)}$$

Equation (1) is homogeneous and is in the form of $Mdx + Ndy = 0$ ... (2)

Comparing equations (1) and (2),

$$M = x^2y - 2xy^2, N = -x^3 + 3x^2y$$

Integrating factor, is expressed as,

$$\text{I.F} = \frac{1}{Mx + Ny} \quad \text{... (3)}$$

Substituting the corresponding values in equation (3),

$$= \frac{1}{x(x^2y - 2xy^2) + (-x^3 + 3x^2y)y}$$

$$= \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2} = \frac{1}{x^2y^2}$$

$$\therefore \text{I.F} = \frac{1}{x^2y^2}$$

Q8. Define linear differential equation.

Answer :

Linear Differential Equation

A differential equation of the form $\frac{dy}{dx} + Py = Q$ (Where P, Q are functions of x or constant) is said to be linear if the dependent variable and its derivatives are of first degree.

Example

$$\frac{dy}{dx} + 2y\tan x = \sin x$$

The general solution of linear differential equation is obtained as,

$$y \times (\text{I.F}) = \int Q \times (\text{I.F})\,d + c$$

Where,

$$\text{Integrating factor (I.F)} = e^{\int P.dx}$$

Q9. Solve $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right)\frac{dx}{dy}=1$

Answer : (Model Paper-3, Q4 | Dec.-13, Q2)

Given differential equation is,

$$\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right)\frac{dx}{dy} = 1 \quad \text{... (1)}$$

Equation (1) can be written as $\frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \quad \text{... (2)}$$

Equation (2) is a linear differential equation of the form,

$$\frac{dy}{dx}+Py = Q \quad \text{... (3)}$$

Comparing equations (2) and (3),

$$P = \frac{1}{\sqrt{x}} \quad Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

Integrating factor (I.F) = $e^{\int Pdx} = e^{\int\left(\frac{1}{\sqrt{x}}\right)dx}$

$\therefore$ I.F = $e^{2\sqrt{x}}$

$\therefore$ The general solution is,

$$y(\text{I.F}) = \int(Q\times(I.F))dx+c \quad \text{... (4)}$$

Substituting the corresponding values in equation (3),

$$y.\,e^{2\sqrt{x}} = \int\frac{e^{-2\sqrt{x}}}{\sqrt{x}}.e^{2\sqrt{x}}\ dx + c$$

$$y.\,e^{2\sqrt{x}} = \int\frac{e^{-2\sqrt{x}+2\sqrt{x}}}{\sqrt{x}}.dx+c$$

$$= \int\frac{e^0}{\sqrt{x}}dx+c$$

$$= \int\frac{1}{\sqrt{x}}dx+c = 2\sqrt{x}$$

$\therefore \quad y.\,e^{2\sqrt{x}} = 2\sqrt{x}$ is the required general solution.

Q10. Define Bernoulli's equation.

Answer : Model Paper-1, Q4

If P and Q are the functions of a variable 'x', then the Bernoulli's equation is defined as,

$$\frac{dy}{dx}+Py = Qy^n$$

Where, n is real constant.

Q11. Define Riccati's equation.

Answer :

A first order differential equation of the form $y' = P(x)y^2 + Q(x)y + R(x)$ is called Riccati's equation.

Q12. Define Clairaut's equation.

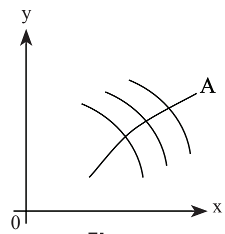
Answer :

An equation of the form $y = px + f(p)$ is called clairaut's equation.

Q13. What is an orthogonal trajectory?

Answer :

A curve intersecting every member of the family of curves at right angle is referred to as orthogonal trajectory and is depicted in figure below.



Figure

Here, curve 'A' represents an orthogonal trajectory.

Q14. Find the orthogonal trajectories of the family of curves $x^2 + y^2 = a^2$.

Answer : Model Paper-2, Q4

Given equation of family of curves,

$$x^2 + y^2 = a^2 \quad \text{... (1)}$$

Differentiating (1) with respect to 'x',

$$2x+2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

The equation of orthogonal trajectories is obtained by replacing $\frac{dy}{dx}$ by $\frac{-dx}{dy}$ and then integrating,

$$-\frac{dx}{dy} = \frac{-x}{y} \quad \Rightarrow \frac{dx}{x} = \frac{dy}{y}$$

Integrating on both sides,

$$\int\frac{1}{x}dx = \int\frac{1}{y}dy$$

$\Rightarrow \quad \log x = \log y + \log c$

$\Rightarrow \quad \log x - \log y = \log c$

$\Rightarrow \quad \log\left(\frac{x}{y}\right) = \log c$

$\therefore \quad \frac{x}{y} = c$

PART-B

ESSAY QUESTIONS WITH SOLUTIONS

2.1 EXACT DIFFERENTIAL EQUATIONS

Q15. What is exact differential equation? Mention the steps involved in determining an exact differential equation.

Answer :

Exact Differential Equation

An equation of the form $Mdx + Ndy = 0$ (where M and N are functions of x and y) is said to be Exact differential equation.

i.e., $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Example

$(x^2 + y)dx + (y^2 + x)dy = 0$

The general solution of this form is obtained as,

$$\int_{(y \text{ constant})} Mdx + \int_{(\text{terms independent of } x)} Ndy = c$$

Procedure

The sequence of steps involved in determining an exact differential equation are:

Step 1

First step is to write the given equation in the form $Mdx + Ndy = 0$.

Step 2

In this step, the differential equation is tested for exactness.

i.e., $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Step 3

The final step is to determine the general solution using the formula,

$$\int_{(y \text{ constant})} Mdx + \int_{(\text{terms independent of } x)} Ndy = c$$

Q16. Solve (x + y – 2)dx + (x – y + 4)dy = 0.

Answer :

Given differential equation is,

$(x + y - 2)dx + (x - y + 4)dy = 0$ ... (1)

Equation (1) is of the form, $Mdx + Ndy = 0$ ... (2)

Comparing equations (1) and (2),

$M = x + y - 2$ ... (3)

$N = x - y + 4$ ... (4)

Partially differentiating equation (3), with respect to 'y',

$\frac{\partial M}{\partial y} = 1$ ... (5)

Partially differentiating equation (4), with respect to x,

$\frac{\partial N}{\partial x} = 1$ .... (6)

From equation (5) and (6),

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Thus, the given differential equation is exact.

The general solution of an exact differential equation is given as,

$$\underset{(y \text{ constant})}{\int} Mdx + \underset{(\text{terms independent of } x)}{\int} Ndy = c$$

$$\int (x+y-2)\,dx + \int (-y+4)\,dy = c$$

$$\Rightarrow \left(\frac{x^2}{2} + xy - 2x\right) + \left(\frac{-y^2}{2} + 4y\right) = c$$

$$\Rightarrow \frac{x^2}{2} - \frac{y^2}{2} + xy - 2x + 4y = c$$

$$\Rightarrow x^2 - y^2 + 2xy - 4x + 8y = 2c$$

$$\therefore x^2 - y^2 + 2xy - 4x + 8y = 2c$$

Q17. Solve the initial value problem $3x^2y^4dx + 4x^3y^3dy = 0$, $y(1) = 2$.

Answer : **(Model Paper-1, Q12(a) | Jan.-12, Q11(a))**

Given differential equation is,

$$3x^2y^4\,dx + 4x^3y^3\,dy = 0 \quad \ldots (1)$$

$$y(1) = 2$$

Equation (1) is of the form,

$$Mdx + Ndy = 0 \quad \ldots (2)$$

Comparing eqations (1) and (2),

$$M = 3x^2y^4, N = 4x^3y^3$$

$$\Rightarrow \frac{\partial M}{\partial y} = 12x^2y^3 \quad \ldots (3)$$

$$\Rightarrow \frac{\partial N}{\partial x} = 12x^2y^3 \quad \ldots (4)$$

From equations (3) and (4),

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence, equation (1) is an exact differential equation

$\therefore$ The general solution is, $\int (Mdx + Ndy) = c$

$$\Rightarrow \int 3x^2y^4dx + \int 4x^3y^3dy = c$$

$$\Rightarrow 3y^4\int x^2dx + 4x^3\int y^3dy = c$$

$$\Rightarrow 3y^4\left(\frac{x^3}{3}\right) + 4x^3\left(\frac{y^4}{4}\right) = c$$

$$\Rightarrow x^3y^4 + x^3y^4 = c$$

$$\Rightarrow 2x^3y^4 = c \quad \ldots (2)$$

Applying initial value condition i.e., $y(1) = 2 \Rightarrow y(x_0) = y_0$

i.e., $x_0 = 1$ and $y_0 = 2$

$\therefore$ $y = 2$ when $x = 1$

Substituting the corresponding values in equation (5),

$$2(1)^3(2)^4 = c$$

$$\therefore 32 = c$$

Substituting the value of c in equation (5),

$$2x^3y^4 = 32 \Rightarrow x^3y^4 = 16$$

$$\therefore x^3y^4 = 16$$

2.2 INTEGRATING FACTORS

Q18. Define integrating factor. Write the rules for finding integrating factor for Mdx + Ndy = 0.

Answer :

Integrating Factor

For answer refer Unit-2, Q5.

Rules for Finding Integrating Factors of the equation Mdx + Ndy = 0

1. Integrating Factor Found by Inspection

In many cases, the integrating factor can be determined by regrouping of terms and recognizing each group as a part of an exact differential equation. The important useful differentials are,

(i) $xdy + ydx = d(xy)$

(ii) $\dfrac{xdy - ydx}{x^2} = d\left(\dfrac{y}{x}\right)$

(iii) $\dfrac{xdy - ydx}{xy} = d\left(\log\left(\dfrac{y}{x}\right)\right)$

(iv) $\dfrac{xdy - ydx}{y^2} = -d\left(\dfrac{x}{y}\right)$

(v) $\dfrac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$

(vi) $\dfrac{xdy - ydx}{x^2 - y^2} = d\left(\dfrac{1}{2}\log\dfrac{x+y}{x-y}\right)$

2. Integrating Factor of a Homogeneous Equation

If $M(x, y)\,dx + N(x, y)\,dy = 0$ represents a homogeneous equation and $Mx + Ny \neq 0$, then the integrating factor of $Mdx + Ndy = 0$ is, $\dfrac{1}{Mx - Ny}$

3. Integrating Factor for an Equation of the Type $f_1(xy)$ y dx + $f_2(xy)$ x dy = 0

If differential equation $Mdx + Ndy = 0$ is of the form $yf_1(xy)\,dx + xf_2(xy)\,dy = 0$ and $Mx - Ny \neq 0$, then integrating factor is given as, $\frac{1}{Mx+Ny}$

4. Integrating Factor of Mdx + Ndy = 0

Case (i)

For a continuous single variable function $f(x)$ such that $\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x} = Nf(x)$, the integrating factor of $Mdx + Ndy = 0$, is given as $e^{\int f(x)dx}$

Case (ii)

For a continuous and differentiable single variable function of $g(y)$ such that $\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y} = Mg(y)$, the integrating factor is, $e^{\int g(y)dy}$

5. Integrating Factor of $x^a y^b$(mydx + nxdy) + $x^{a'} y^{b'}$ (m' ydx + n' xdy) = 0

For the equation $x^a y^b (mydx + nxdy) + x^{a'} y^{b'} (m' ydx + n' xdy) = 0$ an integrating factor is given as $x^h y^k$

Where,

$$\frac{a+h+1}{m} = \frac{b+k+1}{n}, \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$$

Q19. Solve y(2xy + e^x) dx = e^x dy.

Answer :

Given equation is,

$$y(2xy + e^x)\,dx = e^x dy$$

$$\Rightarrow 2xy^2dx + ye^xdx = e^xdy$$

$$\Rightarrow (ye^xdx - e^xdy) + 2xy^2\,dx = 0 \quad \text{... (1)}$$

Dividing equation (1) with y^2,

$$\Rightarrow \frac{(ye^x dx - e^x dy) + 2xy^2 dx}{y^2} = 0$$

$$\Rightarrow \frac{(ye^x dx - e^x dy)}{y^2} + 2x\,dx = 0$$

$$\Rightarrow d\left(\frac{e^x}{y}\right) + 2x\,dx = 0 \quad \text{... (2)}$$

$$\left[\because d\left(\frac{e^x}{y}\right) = \frac{ye^x dx - e^x dy}{y^2}\right]$$

Integrating equation (2),

$$\int d\left(\frac{e^x}{y}\right) + \int 2x dx = 0$$

$$\Rightarrow \frac{e^x}{y} + 2\left[\frac{x^2}{2}\right] = c$$

$\therefore$ The required solution is, $\frac{e^x}{y} + x^2 = c$

Q20. Solve (x³ + y³ + 1)dx + xy²dy = 0.

Answer : June-10, Q2

Given equation is,

$$(x^3 + y^3 + 1)\,dx + xy^2\,dy = 0 \quad \text{... (1)}$$

Equation (1) is of the form $M\,dx + N\,dy = 0$

Comparing equations (1) and (2),

$$M = x^2 + y^3 + 1,\ N = xy^2$$

$$\Rightarrow \frac{\partial M}{\partial y} = 3y^2,\ \frac{\partial N}{\partial x} = y^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Equation (1) is not exact.

Consider,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xy^2}(3y^2 - y^2)$$

$$= \frac{1}{xy^2}(2y^2) = \frac{2}{x}$$

Integrating factor,

$$\text{I.F} = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx}$$

$$= e^{\int \frac{2}{x}dx} = e^{2\log x}$$

$$= e^{\log x^2} = x^2$$

$\therefore$ I.F $= x^2$

Multiply equation (1) by I.F to make it exact.

$$(x^5 + x^2y^3 + x^2)\,dx + x^3\,y^2dy = 0$$

$\therefore$ It is exact differential equation of the form, $Mdx + Ndy = 0$

$\therefore$ The general solution is,

$$\underset{(y\,\text{constant})}{\int Mdx} + \underset{(\text{terms independent of } x)}{\int Ndy} = \text{c}$$

$$\Rightarrow \int [x^5 + x^2y^3 + x^2]\,dx = c$$

$$\Rightarrow \int x^5dx + y^3\int x^2dx + \int x^2dx = c$$

$$\Rightarrow \frac{x^6}{6} + y^3\frac{x^3}{3} + \frac{x^3}{3} = c$$

$\therefore$ The required solution is, $\frac{x^6 + 2x^3y^3 + 2x^3}{6} = c$

Q21. Solve $(3x^2y^3e^y + y^3 + y^2)dx + (x^3y^3e^y - xy)dy = 0$.

Answer : **(Model Paper-2, Q12(a) | June-13, Q11(a) | June-10, Q11(a))**

Given equation is,

$$(3x^2y^3e^y + y^3 + y^2)dx + (x^3y^3e^y - xy)dy = 0 \quad \text{... (1)}$$

Equation (1) is of the form,

$$M\,dx + N\,dy = 0 \quad \text{... (2)}$$

Comparing equations (1) and (2),

$$M = 3x^2y^3e^y + y^3 + y^2 \text{ and } N = x^3y^3e^y - xy$$

$$\frac{\partial M}{\partial y} = e^y(3x^2y^3 + 9x^2y^2) + 3y^2 + 2y$$

$$\frac{\partial N}{\partial x} = 3x^2y^3e^y - y$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ Equation (1) is not exact.

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3x^2y^3e^y + 9x^2y^2e^y + 3y^2 + 2y - 3x^2y^3e^y + y$$

$$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 9x^2y^2e^y + 3y^2 + 3y$$

$$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3(3x^2y^2e^y + y^2 + y)$$

$$\Rightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -3(3x^2y^2e^y + y^2 + y)$$

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{-3(3x^2y^2e^y + y^2 + y)}{3x^2y^3e^y + y^3 + y^2}$$

$$= \frac{-3(3x^2y^2e^y + y^2 + y)}{y(3x^2y^2e^y + y^2 + y)} = \frac{-3}{y}$$

$$\Rightarrow \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{-3}{y}$$

$\therefore$ Integrating factor, I.F $= e^{\int \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dy}$

$$= e^{\int \frac{-3}{y}dy}$$

$$= e^{-3\log y} = e^{\log y^{-3}} = y^{-3}$$

$$\therefore \text{I.F} = \frac{1}{y^3}$$

Multiplying equation (1) by I.F, exact differential equation is obtained as,

$$\left[\frac{3x^2y^3e^y}{y^3} + \frac{y^3}{y^3} + \frac{y^2}{y^3}\right]dx + \left[\frac{x^3y^3e^y}{y^3} - \frac{xy}{y^3}\right]dy = 0$$

$$\Rightarrow \left[3x^2e^y + 1 + \frac{1}{y}\right]dx + \left[x^3e^y - \frac{x}{x^2}\right]dy = 0$$

The above equation is an exact differential equation of the form, $Mdx + Ndy = 0$

$$M = 3x^2e^y + 1 + \frac{1}{y},\ N = x^3e^y - \frac{x}{y^2}$$

$\therefore$ The general solution is,

$$\Rightarrow \underset{(y\text{ constant})}{\int Mdx} + \underset{(\text{terms independent of } x)}{\int Ndy} = c \quad \text{... (3)}$$

Substituting the corresponding values in equation (3),

$$\Rightarrow \int\left[3e^y.x^2 + 1 + \frac{1}{y}\right]dx = c$$

$$\Rightarrow 3e^y\frac{x^3}{3} + x + \frac{x}{y} = c$$

$\therefore$ The required general solution is,

$$x^3e^y + x + \frac{x}{y} = c$$

Q22. Solve $y(xy + 2x^2y^3)\,dx + x(xy - x^2y^2)dy = 0$.

Answer :

Given equation is,

$$y(xy + 2x^2y^3)dx + x(xy - x^2y^2)dy = 0 \quad \text{... (1)}$$

$$xy(ydx + xdy) + x^2y^2(2ydx - xdy) = 0 \quad \text{... (2)}$$

Equation (2) is of the form,

$$x^ax^b(mydx + nxdy) + x^{a'}y^{b'}(m'ydx + n'xdy) = 0 \quad \text{... (3)}$$

Comparing equations (2) and (3),

$a = b = 1$, $m = n = 1$, $a' = b' = 2$, $m' = 2$, $n' = -1$

The integrating factor is given as,

$$x^h.y^k \quad \text{... (4)}$$

Where,

$$\frac{a+h+1}{m} = \frac{b+k+1}{n}, \frac{a'+h+1}{m'} = \frac{b'+k+1}{n'}$$

$$\Rightarrow \frac{1+h+1}{1} = \frac{1+k+1}{1}, \frac{2+h+1}{2} = \frac{2+k+1}{-1}$$

$\Rightarrow h + 2 = k + 2$

$\Rightarrow h - k = 0$

$\Rightarrow h = k$ ... (5)

and $(h + 3)(-1) = (k + 3)2$

$-h - 3 = 2k + 6$

$\Rightarrow -h - 2k - 9 = 0$

$\Rightarrow h + 2k + 9 = 0$ ... (6)

Substituting equation (5) in equation (6),

$k + 2k + 9 = 0$

$\Rightarrow 3k + 9 = 0$

$\Rightarrow 3k = -9$

$\Rightarrow k = -3$

$\therefore h = k = -3$

Substituting the corresponding values in equation (4),

$$I.F = x^{-3}y^{-3}$$

$$\therefore \quad I.F = \frac{1}{x^3y^3} \quad \text{... (7)}$$

Multiplying equation (1) by equation (7),

$$\frac{(xy^2 + 2x^2y^4)dx + (x^2y - x^3y^2)dy}{x^3y^3} = 0$$

$$\Rightarrow \frac{xy^2}{x^3y^3}dx + \frac{2x^2y^4}{x^3y^3}dx + \frac{x^2y}{x^3y^3}dy - \frac{x^3y^2}{x^3y^3}dy = 0$$

$$\Rightarrow \frac{1}{x^2y}dx + \frac{2y}{x}dx + \frac{1}{xy^2}dy - \frac{1}{y}dy = 0$$

$$\Rightarrow \left(\frac{1}{x^2y} + \frac{2y}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0 \quad \text{... (8)}$$

Equation (8) is an exact differential equation of the form $Mdx + Ndy = 0$

$$M = \frac{1}{x^2y} + \frac{2y}{x} \qquad N = \frac{1}{xy^2} - \frac{1}{y}$$

$\therefore$ The general solution is

$$\underset{(y\,\text{constant})}{\int Mdx} + \underset{(\text{terms independent of } x)}{\int Ndy} = c$$

Substituting the corresponding values in equation (9),

$$\Rightarrow \int\left(\frac{1}{x^2y} + \frac{2y}{x}\right)dx + \int\left(-\frac{1}{y}\right)dy = c$$

$$\Rightarrow \frac{1}{y}\int\frac{1}{x^2}dx + 2y\int\frac{1}{x}dx - \int\frac{1}{y}dy = c$$

$$\Rightarrow \frac{1}{y}\int x^{-2}dx + 2y\log x - \log y = c$$

$$\Rightarrow \frac{1}{y}\left(\frac{x^{-1}}{-1}\right) + 2y\log x - \log y = c$$

$$\Rightarrow -\frac{1}{xy} + 2y\log x - \log y = c$$

$\therefore$ The general solution is,

$$2y\log x - \log y - \frac{1}{xy} = c$$

Q23. Solve $(x^3 - 2y^2)dx + 2xy\, dy = 0$.

Answer : **(May/June-17, Q1 | May/June-15, Q11(a))**

Given differential equation is,

$$(x^3 - 2y^2)\, dx + 2xy\, dy = 0 \quad \text{... (1)}$$

Equation (1) is of the form,

$$Mdx + Ndy = 0 \quad \text{... (2)}$$

Comparing equations (1) and (2),

$$M = x^3 - 2y^2 \quad \text{... (3)}$$

And $N = 2xy$ ... (4)

Partially differentiating equation (3) and (4) with respect to 'y' and 'x' respectively,

$$\Rightarrow \frac{\partial M}{\partial y} = -4y, \frac{\partial N}{\partial x} = 2y$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Equation (1) is not exact

Consider,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2xy}(-4y - 2y) = \frac{1}{2xy}(-6y)$$

$$= \frac{-3}{x}$$

$$\text{Integrating factor (I.F)} = e^{\int \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dx}$$

$$= e^{\int \frac{-3}{x}dx}$$

$$= e^{-3\log x}$$

$$= e^{\log x^{-3}} = x^{-3}$$

$$\therefore \text{I.F} = \frac{1}{x^3}$$

Multiplying equation (1) by $I.F$ on both sides,

$$(x^3 - 2y^2)\left(\frac{1}{x^3}\right)dx + 2xy\left(\frac{1}{x^3}\right)dy = 0$$

$$\Rightarrow \left(1 - \frac{2y^2}{x^3}\right)dx + \left(\frac{2y}{x^2}\right)dy = 0 \quad \text{... (5)}$$

Equation (5) is in the form, $M_1dx + N_1dy = 0$

$$M_1 = 1 - \frac{2y^2}{x^3} \text{ and } N_1 = \frac{2y}{x^2}$$

$$\Rightarrow \frac{\partial M_1}{\partial y} = \frac{-4y}{x^3} \text{ and } \frac{\partial N_1}{\partial x} = \frac{-4y}{x^3}$$

$$\Rightarrow \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

$\therefore$ Equation (5) is exact and the general solution is given as,

$$\underset{(y\,\text{constant})}{\int M_1dx} + \underset{(\text{terms independent of } x \text{ in } N_1)}{\int N_1dy} = c$$

Substituting the corresponding values in above equation,

$$\int\left(1 - \frac{2y^2}{x^3}\right)dx + \int 0dy = c$$

$$\Rightarrow \int 1dx - 2\int\frac{y^2}{x^3}dx = c$$

$$\Rightarrow \quad x - 2y^2\left(\frac{x^{-3+1}}{-3+1}\right) = c$$

$$\Rightarrow \quad x - 2y^2\left(\frac{x^{-2}}{-2}\right) = c$$

$$\Rightarrow \quad \frac{x+y^2}{x^2} = c$$

$$\Rightarrow \quad x + y^2 = cx^2$$

$\therefore$ $x + y^2 = cx^2$ is the required solution.

Q24. Solve $(x^2y - 2xy^2)dx + (3x^2y - x^3)dy = 0$.

Answer : April-16, Q11(b)

Given differential equation is,

$$(x^2y - 2xy^2)\,dx + (3x^2y - x^3)\,dy = 0 \qquad \text{... (1)}$$

Equation (1) is a homogenous exact differential equation of the form $Mdx + Ndy = 0$.

$$M = x^2y - 2xy^2 \text{ and } N = 3x^2y - x^3$$

Hence, the integrating factor is given as

$$I.F = \frac{1}{Mx+Ny}$$

$$= \frac{1}{(x^2y - 2xy^2)x + y(3x^2y - x^3)}$$

$$= \frac{1}{x^3y - 2x^2y^2 + 3x^2y^2 - x^3y}$$

$$\therefore \quad I.F = \frac{1}{x^2y^2}$$

Multiply equation (1) with integrating factor,

$$\frac{1}{x^2y^2}\left[(x^2y - 2xy^2)\,dx + (3x^2y - x^3)\,dy\right] = 0$$

$$\Rightarrow \quad \frac{x^2y - 2xy^2}{x^2y^2}dx + \frac{3x^2y - x^3}{x^2y^2}dy = 0$$

$$\Rightarrow \quad \left(\frac{x^2y}{x^2y^2} - \frac{2xy^2}{x^2y^2}\right)dx + \left(\frac{3x^2y}{x^2y^2} - \frac{x^3}{x^2y^2}\right)dy = 0$$

$$\therefore \quad \left(\frac{1}{y} - \frac{2}{x}\right)dx + \left(\frac{3}{y} - \frac{x}{y^2}\right)dy = 0 \qquad \text{... (2)}$$

Equation (2) is in the form of exact differential equation $M_1dx + N_1dy = 0$

Where,

$$M_1 = \frac{1}{y} - \frac{2}{x}\ , \quad N_1 = \frac{3}{y} - \frac{x}{y^2}$$

$$\frac{\partial M_1}{\partial y} = -\frac{1}{y^2} - 0\ , \quad \frac{\partial N_1}{\partial x} = 0 - \frac{1}{y^2}$$

$$= \frac{-1}{y^2}\ , \qquad = \frac{-1}{y^2}$$

$$\therefore \quad \frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

Hence, equation (2) is exact and its solution is given as,

$$\underset{(y\,\text{constant})}{\int M_1dx} + \underset{(\text{terms independent of } x)}{\int N_1dy} = c$$

$$\Rightarrow \quad \int\left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}dy = c$$

$$\Rightarrow \quad \int \frac{1}{y}dx - \int \frac{2}{x}dx + 3\int \frac{1}{y}dy = c$$

$$\Rightarrow \quad \frac{1}{y}\int 1dx - 2\int \frac{1}{x}dx + 3\int \frac{1}{y}dy = c$$

$$\Rightarrow \quad \frac{1}{y}(x) - 2(\log x) + 3(\log y) = c$$

$\therefore$ The general solution is given as,

$$\frac{x}{y} - 2\log x + 3\log y = c$$

2.3 LINEAR DIFFERENTIAL EQUATIONS

Q25. Define linear differential equations. Mention the steps involved in determining the linear equations.

Answer :

Linear Differential Equation

For answer refer Unit-2, Q8.

Procedure

Step 1

Write the given equation in the form $\frac{dy}{dx} + Py = Q$

Step 2

In this step, the functions P and Q are identified.

Step 3

Evaluate the integrating factor i.e., $IF = e^{\int P.dx}$

Step 4

Determine the general solution using the formula,

$$y \times (\text{I.F}) = \int Q \times (I.F)dx + c$$

Q26. Solve $\frac{dy}{dx} + y\tan x = \sec x$.

Answer :

Given differential equation is,

$$\frac{dy}{dx} + y\tan x = \sec x$$

The above equation is of form, $\frac{dy}{dx} + yP(x) = Q(x)$, which is a linear differential equation.

Where,

$$P(x) = \tan x,\ Q(x) = \sec x$$

Integrating factor is expressed as,

$$\text{I.F} = e^{\int P(x)dx} = e^{\int \tan x dx}$$

$$= e^{\log_e \sec x}$$

$\therefore$ I.F = sec x

$\therefore$ The general solution is

$$y(\text{I.F}) = \int (\text{I.F.})Q(x)dx + c$$

Substituting the corresponding values in above equation,

$$y(\sec x) = \int \sec x.\sec x\, dx + c$$

$$y(\sec x) = \int \sec^2 x\, dx + c$$

$$y(\sec x) = \tan x + c$$

$$y = \frac{\tan x}{\sec x} + \frac{c}{\sec x}$$

$\therefore$ $y = \sin x + c\cos x$

Q27. Solve $x\dfrac{dy}{dx} + y = x^3 y^6$.

Answer : May/June-12, Q11(a)

Given differential equation is,

$$x\frac{dy}{dx} + y = x^3 y^6 \quad \text{... (1)}$$

Dividing equation (1) by xy^6 on both sides,

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{xy^5} = x^2 \quad \text{... (2)}$$

Let, $\dfrac{1}{y^5} = v$

Differentiating on both sides with respect to x,

$$\frac{-5}{y^6}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \frac{1}{y^6}\frac{dy}{dx} = \frac{-1}{5}\frac{dv}{dx} \quad \text{... (3)}$$

Substituting equation (3) and 'v' in equation (2),

$$\frac{-1}{5}\frac{dv}{dx} + \frac{v}{x} = x^2$$

$$\Rightarrow \frac{dv}{dx} - \frac{5v}{x} = -5x^2$$

The above equation is in the form of $\dfrac{dy}{dx} + P(x)y = Q(x)$

Where, $P(x) = \dfrac{-5}{x}$, $Q(x) = -5x^2$

Integrating factor I.F $= e^{\int P(x)dx}$

$$= e^{\int \frac{-5}{x}dx} = e^{-5\int \frac{1}{x}dx}$$

$$= e^{-5\log x}$$

$$= e^{\log x^{-5}}$$

$$= \frac{1}{x^5}$$

The solution of the given differential equation is,

$$v \times \text{I.F} = \int (\text{I.F}) \times Q(x)\, dx$$

Substituting the corresponding values in above equation,

$$\frac{1}{y^5} \times \frac{1}{x^5} = \int \left(\frac{1}{x^5} \times -5x^2\right) dx$$

$$\Rightarrow \frac{1}{y^5 x^5} = -5\int \frac{1}{x^3}dx$$

$$\Rightarrow \frac{1}{y^5 x^5} = -5 \times \frac{-1}{2x^2} + c$$

$$\Rightarrow \frac{1}{y^5 x^5} = \frac{5}{2x^2} + c$$

$\therefore$ The general solution is given as,

$$\frac{1}{y^5 x^5} = \frac{5}{2x^2} + c$$

Q28. Solve $\dfrac{dy}{dx} - y = y^2(\sin x + \cos x)$.

(June-14, Q11(b) | June-13, Q11(b) | Jan.-12, Q11(b))

OR

Solve the differential equation,

$$\frac{dy}{dx} - y = y^2(\sin x + \cos x).$$

Answer : Model Paper-3, Q12(a)

Given differential equation is,

$$\frac{dy}{dx} - y = y^2(\sin x + \cos x)$$

Dividing with y^2 on both sides,

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = \sin x + \cos x \quad \text{... (1)}$$

$$\frac{1}{y^2}\frac{dy}{dx} + (1)\left(\frac{-1}{y}\right) = \sin x + \cos x$$

Let, $\dfrac{-1}{y} = v$

Differentiating on both sides w.r.t. 'x',

$$\frac{1}{y^2}\frac{dy}{dx} = \frac{dv}{dx} \quad \text{... (2)}$$

Substituting equation (2) in equation (1),

$$\frac{dv}{dx} + (1)v = \sin x + \cos x$$

The above equation is in the form of, $\frac{dy}{dx} + P(x)y = Q(x)$

Where,

$$P(x) = 1,\ Q(x) = \sin x + \cos x$$

Integrating factor = $e^{\int P(x)dx}$

$$= e^{\int 1.dx} = e^x$$

The solution of differential equation is,

$$v(\text{I.F}) = \int (I.F)Q(x)dx + c$$

Substituting the corresponding values in above equation,

$$v.e^x = \int e^x(\sin x + \cos x)dx$$

$$\left(\frac{-1}{y}\right)e^x = \int e^x \sin x dx + \int e^x \cos x dx$$

$$= \frac{e^x}{1^2+1^2}(\sin x - \cos x) + \frac{e^x}{1^2+1^2}(\cos x + \sin x) + c$$

$$\left[\because \int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx),\ \int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$\Rightarrow \quad -\frac{e^x}{y} = \frac{e^x}{2}(\sin x - \cos x) + \frac{e^x}{2}(\cos x + \sin x) + c$$

$$\Rightarrow \quad -\frac{2e^x}{y} = e^x(\sin x - \cos x) + e^x(\cos x + \sin x) + 2c$$

$$\Rightarrow \quad -\frac{2}{y} = \sin x - \cos x + \sin x + \cos x + 2ce^{-x}$$

$$y = \frac{-2}{(\sin x - \cos x) + (\cos x + \sin x) + ce^{-x}}$$

$$= \frac{-2}{2\sin x + 2ce^{-x}}$$

$\therefore$ The general solution is $y = \frac{-1}{\sin x + ce^{-x}}$

Q29. Solve the differential equation, $\cot 3x \frac{dy}{dx} - 3y = \cos 3x + \sin 3x$.

Answer : Dec.-12, Q11(a)

Given differential equation is,

$$\cot 3x \frac{dy}{dx} - 3y = \cos 3x + \sin 3x$$

Dividing both the sides by cot 3x,

$$\frac{dy}{dx} + \left(\frac{-3}{\cot 3x}\right)y = \frac{\cos 3x + \sin 3x}{\cot 3x}$$

$$\Rightarrow \quad \frac{dy}{dx}+(-3\tan 3x)y = \frac{\cos 3x}{\cot 3x}+\frac{\sin 3x}{\cot 3x}$$

$$= \frac{\cos 3x}{\left(\frac{\cos 3x}{\sin 3x}\right)}+\frac{\sin 3x}{\left(\frac{\cos 3x}{\sin 3x}\right)}$$

$$= \sin 3x+\frac{\sin^2 3x}{\cos 3x}$$

$$= \sin 3x + \sin 3x \tan 3x$$

$$\Rightarrow \quad \frac{dy}{dx}+(-3\tan 3x)y = \sin 3x(1 + \tan 3x) \quad \text{... (1)}$$

Equation (1) represents a linear differential equation of the form,

$$\frac{dy}{dx}+Py=Q \quad \text{... (2)}$$

Comparing equations (1) and (2),

$P = -3 \tan 3x$, $Q = \sin 3x(1 + \tan 3x)$

Integrating factor (I.F) is $= e^{\int P dx}$

$$= e^{\int(-3\tan 3x)dx} = e^{(-1)\int 3\tan 3x dx}$$

$$= e^{(-1)\log_e \sec 3x} = e^{\log_e (\sec 3x)^{-1}}$$

$$= (\sec 3x)^{-1} = \cos 3x$$

$\therefore$ I.F = $\cos 3x$

$\therefore$ The general solution of the given differential equation is obtained as,

$$y(I.F) = \int(I.F \times Q)\, dx + c$$

Substitution the corresponding values in above equation,

$$\Rightarrow \quad y\cos 3x = \int[\cos 3x \times \sin 3x\,(1 + \tan 3x)]\, dx + c$$

$$= \int[\cos 3x \times (\sin 3x + \sin 3x \tan 3x)]\, dx + c$$

$$= \int[\sin 3x \cos 3x + \sin 3x \cos 3x \tan 3x]\, dx + c$$

$$= \int \sin 3x \cos 3x\, dx + \int \sin 3x \cos 3x \times \frac{\sin 3x}{\cos 3x} dx + c$$

$$= \int \frac{\sin 2(3x)}{2} dx + \int \sin^2 3x dx + c \qquad (\because \sin 2\theta = 2\sin\theta\cos\theta)$$

$$= \frac{1}{2}\int \sin 6x dx + \int\left(\frac{1-\cos 2(3x)}{2}\right)dx + c \qquad \left(\because \sin^2\theta = \frac{1-\cos 2\theta}{2}\right)$$

$$= \frac{1}{2}\left(\frac{-\cos 6x}{6}\right)+\frac{1}{2}\int(1-\cos 6x)dx + c$$

$$= \frac{-\cos 6x}{12}+\frac{1}{2}\left(x-\frac{\sin 6x}{6}\right)+c$$

$$= \frac{-\cos 6x}{12}+\frac{x}{2}-\frac{\sin 6x}{12}+c$$

$$= \frac{1}{12}(6x - \sin 6x - \cos 6x) + c$$

$\therefore$ The general solution is, $y\cos 3x = \frac{1}{12}(6x - \sin 6x - \cos 6x) + c$

Q30. Solve $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$.

Answer : Dec.-13, Q11(a)

Given differential equation is,

$$\frac{dy}{dx} = e^{x-y}(e^x - e^y) \quad \text{... (1)}$$

$$\Rightarrow \frac{dy}{dx} = e^x.e^{-y}(e^x - e^y)$$

$$\Rightarrow \frac{dy}{dx}.e^y = e^x(e^x - e^y)$$

$$\Rightarrow \frac{dy}{dx}e^y = e^{2x} - (e^x.e^y)$$

$$\Rightarrow e^y\frac{dy}{dx} + e^x.e^y = e^{2x} \quad \text{... (2)}$$

Let,

$$e^y = u$$

Differentiating on both sides with respect to 'x',

$$\Rightarrow e^y\frac{dy}{dx} = \frac{du}{dx}$$

Substituting the value of e^y and $e^y\frac{dy}{dx}$ in equation (2),

$$\frac{du}{dx} + e^x.\, u = e^{2x}$$

The above equation is a linear differential equation of the form, $\frac{dy}{dx} + Py = Q$

$$P = e^x,\ Q = e^{2x}$$

Integrating factor is, $e^{\int Pdx} = e^{\int e^x dx} = e^{e^x}$

$\therefore$ The general solution is,

$$u.(\text{I.F}) = \int Q.(IF)dx + c$$

Substituting the corresponding values in the above equation,

$$\Rightarrow e^y e^{e^x} = \int Q.(IF)dx + c$$

$$\Rightarrow e^y e^{e^x} = \int e^{2x}.e^{e^x}dx + c$$

Let,

$$e^x = t$$

Differentiating on both sides with respect to 'x',

$$\Rightarrow e^x\, dx = dt$$

$$\Rightarrow e^y e^t = \int t\,e^t dt + c = \left[t(e^t) - 1e^t\right] + c$$

$$\Rightarrow e^y.e^t = e^t[t-1] + c$$

$$\Rightarrow e^y.e^{e^x} = e^{e^x}[e^x - 1] + c$$

$$\Rightarrow e^y.e^{e^x} - e^{e^x}(e^x - 1) = c$$

$$\Rightarrow e^{e^x}[e^y - (e^x - 1)] = c$$

$\therefore$ $e^{e^x}(e^y - e^x + 1) = c$ is the general solution.

2.4 BERNOULLI'S, RICCATI'S AND CLAIRAUT'S DIFFERENTIAL EQUATIONS

Q31. Define Bernoulli's equation. Write the steps involved in solving a Bernoulli's equation.

Answer :

For answer refer Unit-2, Q10.

The steps involved in solving a Bernoulli's equation are:

(i) Initially, the given differential equation should be converted into the standard form of Bernoullis equation.

i.e., $\frac{dy}{dx} + Py = Qy^n$

(ii) Divide the entire equation by y^n to obtain an equation of the form, $y^{-n}\frac{dy}{dx} + Py^{1-n} = Q.$

(iii) Replace y^{1-n} by t and solve to obtain a linear equation in 't'.

(iv) Finally, replace t by y^{1-n} in the solution obtained in step (iii) to achieve the desired solution.

Q32. Solve $\frac{dy}{dx}(x^2y^3 + xy) = 1$.

Answer : Model Paper-1, Q12(b)

Given differential equation is,

$$\frac{dy}{dx}(x^2y^3 + xy) = 1$$

$$\Rightarrow dy(x^2y^3 + xy) = dx$$

$$\Rightarrow \frac{dx}{dy} = x^2y^3 + xy$$

$$\Rightarrow \frac{dx}{dy} - xy = x^2y^3 \quad \text{... (1)}$$

Dividing equation (1) by x^2,

$$\frac{1}{x^2}\frac{dx}{dy} - \frac{1}{x}.y = y^3 \quad \text{... (2)}$$

Let , $\frac{1}{x} = z \quad \text{... (3)}$

Differentiating on both sides with respect to y,

$$\frac{dz}{dy} = \frac{-1}{x^2}\frac{dx}{dy} \quad \text{... (4)}$$

$$\Rightarrow \frac{dx}{dy} = -x^2\frac{dz}{dy}$$

Substituting equations (3) and (4) in equation (2),

$$-\frac{dz}{dy} - zy = y^3$$

$$\Rightarrow \frac{dz}{dy} + zy = -y^3 \quad \text{... (5)}$$

Equation (5) is a linear differential equation in 'z' i.e., $\frac{dz}{dy} + Pz = Q$

$$P = y, \quad Q = -y^3 \qquad \text{... (6)}$$

The general solution is,

$$z \times \text{I.F} = \int Q.(I.F)dy + c \qquad \text{... (7)}$$

$$\text{I.F} = e^{\int P.dy} = e^{\int y.dy}$$

$$\text{I.F} = e^{y^2/2} \qquad \text{... (8)}$$

Substituting the corresponding values in equation (7),

$$z \times e^{y^2/2} = \int -y^3(e^{y^2/2})dy + c$$

$$ze^{y^2/2} = -\int y^3.e^{y^2/2}dy \qquad \text{... (9)}$$

Let, $\frac{y^2}{2} = t$

Differentiating the above equation on both sides,

$$ydy = dt$$

The R.H.S of equation (9) becomes

$$-\int e^{y^2/2}.y^3dy = -\int e^t(2t)dt$$

$$= -2(te^t - \int 1.e^t\, dt)$$

$$= -2e^t(t-1) + c$$

$$= -2e^{y^2/2}\left(\frac{y^2}{2} - 1\right) + c$$

Substituting the above value in equation (9),

$$ze^{y^2/2} = -2e^{y^2/2}\left[\frac{y^2}{2} - 1\right] + c$$

(or)

$$\frac{1}{x}e^{y^2/2} = -2e^{y^2/2}\left[\frac{y^2}{2} - 1\right] + c$$

$\therefore$ $\frac{1}{x}e^{y^2/2} = (2-y^2)e^{y^2/2} + c$ is the general solution.

Q33. Define Riccati's equation. Write the procedure for determining the Riccati's equation.

Answer :

Riccati's Equation

For answer refer Unit-2, Q11.

Procedure

Step 1

Initially, the given equation is expressed in the form of,

$$y' = Py^2 + Qy + R$$

Step 2

In this step, the particular solution $v(x)$ is determined.

Step 3

In next step, a linear differential equation (Leibnitz equation) is determined by substituting,

$$y(x) = v(x) + \frac{1}{z(x)}$$

Step 4

Finally the required general solution is obtained.

Q34. Find the general solution of the Riccati equation
y' = 3y² – (1 + 6x) y + 3x² + x + 1, if y = x is a particular solution.

(Model Paper-2, Q12(b) | Dec.-12, Q11(b))

OR

Solve y' = 3y² – (1 + 6x)y + 3x² + x + 1, y = x is particular solution.

Answer :

Given differential equation is,

$$y' = 3y^2 - (1 + 6x)y + 3x^2 + x + 1$$

$$\Rightarrow \quad \frac{dy}{dx} = 3y^2 - (1 + 6x)y + 3x^2 + x + 1 \qquad \text{... (1)}$$

The particular solution is, $y = x$

$$y = v(x)$$

$$\Rightarrow \quad v(x) = x \qquad (\because y = x)$$

Let, $y = v(x) + \frac{1}{z}$

$$\Rightarrow \quad y = x + \frac{1}{z} \qquad \text{... (2)}$$

Differentiating equation (2) with respect to 'x',

$$\frac{dy}{dx} = 1 - \frac{1}{z^2}\frac{dz}{dx} \qquad \text{... (3)}$$

Substituting equation (3) in equation (1),

$$1 - \frac{1}{z^2}\frac{dz}{dx} = 3\left[x + \frac{1}{z}\right]^2 - (1+6x)\left[x + \frac{1}{z}\right] + 3x^2 + x + 1$$

$$\Rightarrow \quad -\frac{1}{z^2}\frac{dz}{dx} = 3\left[x^2 + \frac{1}{z^2} + \frac{2x}{z}\right] - (1 + 6x)x - (1 + 6x)\frac{1}{z} + 3x^2 + x$$

$$\Rightarrow \quad -\frac{1}{z^2}\frac{dz}{dx} = 3x^2 + \frac{3}{z^2} + \frac{6x}{z} - x - 6x^2 - \frac{1}{z} - \frac{6x}{z} + 3x^2 + x$$

$$\Rightarrow \quad -\frac{1}{z^2}\frac{dz}{dx} = \frac{3}{z^2} - \frac{1}{z}$$

$$\Rightarrow \quad \frac{dz}{dx} = z - 3$$

$$\Rightarrow \quad \frac{dz}{dx} - z = -3 \qquad \text{... (4)}$$

The general linear differential equation is,

$$\frac{dy}{dx} + P(x) = Q(x) \qquad \text{... (5)}$$

Comparing equation (4) with equation (5),

$P(x) = -1$, $y = z$ and $Q(x) = -3$

The integral factor is, $e^{\int P dx}$

$$= e^{-\int dx} = e^{-x}$$

The general solution is, $y\,(\text{I.F.}) = \int (\text{I.F})\, Q\, dx + c$

$$z e^{-x} = \int e^{-x}(-3)dx + c$$

$\Rightarrow \quad z\,e^{-x} = 3e^{-x} + c$

$\therefore \quad z = 3 + c\,e^{x}$ ... (6)

Substituting equation (6) in equation (2),

$$y = x + \frac{1}{3+ce^x}$$

$\therefore$ The general solution is $y = x + \dfrac{1}{3+ce^x}$.

Q35. Find the general solution of the Riccati equation y' = 4xy² + (1 – 8x)y + 4x – 1, y = 1 is a particular solution.

Answer : Jan.-12, Q12(a)

Given differential equation is,

$$y' = 4xy^2 + (1-8x)y + 4x - 1$$

$$\Rightarrow \frac{dy}{dx} = 4xy^2 + (1-8x)y + 4x - 1 \quad \text{... (1)}$$

The particular solution is,

$y = 1$

$y = v(x)$

$\therefore \quad v(x) = 1$

Let, $y = v(x) + \dfrac{1}{z}$

$$\Rightarrow y = 1 + \frac{1}{z} \quad \text{... (2)}$$

Differentiating equation (2) w.r.t. 'x',

$$\frac{dy}{dx} = \frac{-1}{z^2}\frac{dz}{dx} \quad \text{... (3)}$$

Substituting equations (2) and (3) in equation (1),

$$\frac{-1}{z^2}\frac{dz}{dx} = 4x\left[1+\frac{1}{z}\right]^2 + (1-8x)\left[1+\frac{1}{z}\right] + 4x - 1$$

$$\Rightarrow \frac{-1}{z^2}\frac{dz}{dx} = 4x\left[1+\frac{1}{z^2}+\frac{2}{z}\right] + (1-8x)\left[1+\frac{1}{z}\right] + 4x - 1$$

$$\Rightarrow \frac{-1}{z^2}\frac{dz}{dx} = 4x + \frac{4x}{z^2} + \frac{8x}{z} + 1 + \frac{1}{z} - 8x - \frac{8x}{z} + 4x - 1$$

$$\Rightarrow \frac{-1}{z^2}\frac{dz}{dx} = \frac{4x}{z^2} + \frac{1}{z}$$

$$\Rightarrow \frac{dz}{dx} = -z^2\left[\frac{4x}{z^2} + \frac{1}{z}\right]$$

$$\Rightarrow \frac{dz}{dx} = -z^2\left[\frac{4x+z}{z^2}\right]$$

$$\Rightarrow \frac{dz}{dx} = -[4x + z]$$

$$\Rightarrow \frac{dz}{dx} = -4x - z$$

$$\therefore \frac{dz}{dx} + z = -4x \quad \text{... (4)}$$

Equation (4) is a linear differential equation of the form,

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \text{... (5)}$$

Comparing equation (4) with equation (5),

$P(x) = 1, \quad y = z \quad$ and $\quad Q(x) = -4x$

The integral factor is, $e^{\int P(x)dx}$

$$= e^{\int 1.dx} = e^x$$

$\therefore \quad$ I.F $= e^x$

$\therefore$ The general solution is,

$$y(\text{I.F}) = \int (I.F).Q(x)dx + c$$

Substituting the corresponding values in the above equation,

$$z.e^x = \int e^x(-4x)dx + c$$

$$\Rightarrow z.e^x = -4\int e^x x\,dx + c$$

$$\Rightarrow z.e^x = -4\left[x.\int e^x - \left[\int 1 \times \int e^x dx\right]dx\right] + c$$

$$\Rightarrow z.e^x = -4[xe^x - e^x] + c$$

$$\Rightarrow z.e^x = -4e^x[x-1] + c$$

$$\Rightarrow z.e^x = 4e^x[1-x] + c$$

$$\Rightarrow z = \frac{4e^x[1-x]+c}{e^x}$$

$$\therefore z = 4[1-x] + ce^{-x} \quad \text{... (6)}$$

Substituting equation (6) in equation (2),

$$y = 1 + \frac{1}{4[1-x]+ce^{-x}}$$

The general solution is, $y = 1 + \dfrac{1}{4[1-x]+ce^{-x}}$

Q36. Define Clairaut's equation. write the Steps involved in determining the Clairaut's equation.

Answer :

Clairut's Equation

For answer refer Unit-2, Q12.

Procedure

Step 1

Initially, the given equation is expressed in the form,

If, $y = px + f(p)$ ... (1)

Step 2

In this step, equation (1) is differentiated with respect to 'x'.

i.e., $\frac{dy}{dx} = \frac{d}{dx}(px) + \frac{d}{dx}f(p)$

$= p\frac{dx}{dx} + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$

$\Rightarrow \frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$

$= \frac{dp}{dx}(x + f'(p)) + p$

$p = \frac{dp}{dx}(x + f'(p)) + p \qquad \left[\because \frac{dy}{dx} = p\right]$

$\Rightarrow \frac{dp}{dx}(x + f'(p)) = 0$

$\Rightarrow \frac{dp}{dx} = 0 \Rightarrow p = c$

Step 3

Substituting, $p = c$ in equation (1),

$y = cx + f(c)$

Thus, the required general solution is obtained.

Q37. Find the general solution and the singular solution of the Clairaut's equation y = xy' + (y')².

Jan.-10, Q11(b)

OR

Solve $y = xy' + (y')^2$.

Answer :

Given differential equation is,

$y = xy' + (y')^2$

$\Rightarrow y = xy' + f(y')$

Where,

$f(y') = (y')^2 \Rightarrow f(c) = (c)^2$

The general solution is,

$y = xc + f(c)$

$y = xc + c^2$ ... (1)

The singular solution is obtained by differentiating equation (1) with respect to 'x',

$\frac{dy}{dx} = x.\frac{dc}{dx} + c\frac{d}{dx}(x) + \frac{d}{dx}(c^2)$

$\left[\because \frac{d}{dx}u.v = uv' + vu'\right]$

$\Rightarrow \frac{dy}{dx} = x.\frac{dc}{dx} + c.1 + 2.c\frac{dc}{dx}$

$\Rightarrow \frac{dy}{dx} = x.\frac{dc}{dx} + c + 2c\frac{dc}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{dc}{dx}(x + 2c) + c$

$\Rightarrow c = \frac{dc}{dx}(x + 2c) + c \qquad \left(\because \frac{dy}{dx} = c\right)$

$\Rightarrow \frac{dc}{dx}(x + 2c) = 0$

$\Rightarrow x + 2c = 0$

$\Rightarrow x = -2c$ ... (2)

Substituting equation (2) in equation (1),

$y = (-2c)(c) + c^2$

$\Rightarrow y = -2c^2 + c^2$

$\Rightarrow y = -c^2$

$\therefore c^2 = -y$ ... (3)

From equation (2) $c = \frac{-x}{2}$

Squaring on both sides,

$c^2 = \frac{x^2}{4}$ ... (4)

Comparing equations (3) and (4),

$-y = \frac{x^2}{4}$

$\Rightarrow -4y = x^2$

$\Rightarrow x^2 + 4y = 0$

$\therefore$ The general solution is, $y = xc + c^2$ and the singular solution is, $x^2 + 4y = 0$.

Q38. Find the general and the singular solution of Clairaut's equation y = xy' – (y')³.

Answer : **(Model Paper-3, Q12(b) | June-13, Q12(a) | June-11, Q12(a))**

Given differential equation is,

$y = xy' - (y')^3$

The above equation is in the form of,

$y = px + f(p)$

$\therefore y = xy' - f(y')$

Where,

$f(y') = (y')^3$

$\Rightarrow f(c) = (c)^3 \qquad (\because y' = c)$

The general solution is, $y = xc + f(c)$

$y = xc - c^3$ ... (1)

The singular solution is obtained by differentiating equation (1) with respect to 'x'.

$$\frac{dy}{dx} = x\frac{dc}{dx} + c\frac{d}{dx}(x) - \frac{d}{dx}(c^3) \qquad \left[\because \frac{d}{dx}u.v = uv' + vu'\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = x.\frac{dc}{dx} + c.1 - 3c^2\frac{dc}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = x.\frac{dc}{dx} + c - 3c^2\frac{dc}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dc}{dx}[x - 3c^2] + c$$

$$\Rightarrow \quad c = \frac{dc}{dx}[x - 3c^2] + c \qquad \left[\because \frac{dy}{dx} = c\right]$$

$$\Rightarrow \quad \frac{dc}{dx}[x - 3c^2] = 0$$

$$\Rightarrow \quad x - 3c^2 = 0$$

$$\Rightarrow \quad x = 3c^2 \qquad \text{... (2)}$$

Substituting equation (2) in equation (1),

$$y = 3c^2.(c) - c^3$$

$$\Rightarrow \quad y = 3c^3 - c^3$$

$$\Rightarrow \quad y = 2c^3$$

$$\Rightarrow \quad c^3 = \frac{y}{2} \qquad \text{.... (3)}$$

From equation (2) $c^2 = \frac{x}{3}$

$$\Rightarrow \quad c = \sqrt{\frac{x}{3}} \qquad \text{... (4)}$$

Substituting the value of c in equation (3),

$$\left(\sqrt{\frac{x}{3}}\right)^3 = \frac{y}{2}$$

Squaring on both sides,

$$\Rightarrow \quad \left[\frac{x}{3}\right]^3 = \left(\frac{y}{2}\right)^2$$

$$\Rightarrow \quad \frac{x^3}{27} = \frac{y^2}{4}$$

$$\Rightarrow \quad y^2 = \frac{4x^3}{27}$$

$\therefore$ The general solution is $y = xc - c^3$ and the singular solution is, $y^2 = \frac{4x^3}{27}$.

2.5 ORTHOGONAL TRAJECTORIES OF A GIVEN FAMILY OF CURVES

Q39. Define orthogonal trajectory and write the procedure to obtain equation of orthogonal trajectory for Cartesian and polar curves.

Answer :

Orthogonal Trajectory

For answer refer Unit-2, Q13.

Procedure to Find Equation of Orthogonal Trajectories of an Cartesian Curves

Consider the Cartesian curve,

$$f(x, y, c) = 0 \quad \text{... (1)}$$

Where,

c is the arbitrary constant.

Step (i)

The first step is to differentiate equation (1) with respect to x,

Step (ii)

In this step, eliminate the arbitrary constant 'c' by substitution method which forms a differential equation of the form.

$$F\left(x, y, \frac{dy}{dx}\right) = 0 \quad \text{... (2)}$$

Step (iii)

By replacing $\frac{dy}{dx}$ with $\frac{-dx}{dy}$ in the equation (2),

$$F\left(x, y, \frac{-dx}{dy}\right) = 0 \quad \text{... (3)}$$

Step (iv)

Finally, by solving equation (3) the required orthogonal trajectories are obtained.

Procedure to Find Equation of Orthogonal Trajectories of Polar Curves

Consider a polar curve,

$$f(r, \theta, c) = 0 \quad \text{... (4)}$$

Where,

c = Arbitrary constant.

Step (i)

Initially differentiate equation (4) with respect to θ,

Step (ii)

In this step, eliminate the arbitrary constant 'c' by substitution method which forms a differential equation of the form,

$$F\left(r, \theta, \frac{dr}{d\theta}\right) = 0 \quad \text{... (5)}$$

Step (iii)

By replacing $\frac{dr}{d\theta}$ with $-r^2\frac{d\theta}{dr}$ in equation (5),

$$f\left(r, \theta, -r^2\frac{d\theta}{dr}\right) = 0 \quad \text{... (6)}$$

Step (iv)

Finally, by solving equation (6) required orthogonal trajectories are obtained.

Q40. Find the orthogonal trajectories of the family of curves r = c(sec θ + tan θ).

Answer : June-10, Q11(b)

Given family of curves,

$$r = c(\sec\theta + \tan\theta) \quad \text{... (1)}$$

Differentiating equation (1) with respect to 'θ',

$$\frac{dr}{d\theta} = c(\sec\theta\tan\theta + \sec^2\theta)$$

$$\Rightarrow c = \frac{1}{\sec\theta(\sec\theta + \tan\theta)} \cdot \frac{dr}{d\theta} \quad \text{... (2)}$$

Substituting equation (2) in equation (1),

$$r = \frac{1}{\sec\theta(\sec\theta + \tan\theta)} \times (\sec\theta + \tan\theta) \cdot \frac{dr}{d\theta}$$

$$= \frac{1}{\sec\theta} \cdot \frac{dr}{d\theta}$$

$$r = \cos\theta \frac{dr}{d\theta}$$

Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

$$\therefore \quad r = -\cos\theta . r^2 \frac{d\theta}{dr}$$

$$\Rightarrow \quad \frac{1}{r}dr = -\cos\theta\, d\theta$$

$$\Rightarrow \quad \frac{1}{r}dr + \cos\theta\, d\theta = 0 \quad \text{... (3)}$$

Integrating both sides of equation (3),

$$\int \frac{1}{r}dr + \int \cos\theta\, d\theta = \log c$$

$$\Rightarrow \quad \log r + \sin\theta = \log c$$

$$\Rightarrow \quad \log r - \log c + \sin\theta = 0$$

$$\Rightarrow \quad \log\left(\frac{r}{c}\right) + \sin\theta = 0$$

∴ The equation of the orthogonal trajectories is $\log\left[\frac{r}{c}\right] + \sin\theta = 0$.

Q41. Find the orthogonal trajectories of r = ce^θ, where c is the parameter.

Answer : June-14, Q11(a))

Given,

$$r = ce^{\theta} \quad \text{... (1)}$$

Differentiating equation (1) with respect to θ,

$$\frac{dr}{d\theta} = c.e^{\theta} \quad \text{... (2)}$$

$$\Rightarrow \quad c = \frac{dr}{d\theta}.e^{-\theta} \quad \text{... (3)}$$

Substituting equation (3) in equation (1),

$$r = \left[\frac{dr}{d\theta}.e^{-\theta}\right]e^{\theta}$$

$$\Rightarrow \quad r = \frac{dr}{d\theta}$$

This is the differential equation of the family of curves.

Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

$$\Rightarrow \quad r = \left(-r^2\frac{d\theta}{dr}\right)$$

$$r = -r^2\frac{d\theta}{dr}$$

$$\frac{1}{r}dr = -d\theta$$

Integrating on both sides

$$\int \frac{1}{r}dr = -\int d\theta$$

$$\log r = -\theta + \log c$$

$$\Rightarrow \quad \log\left(\frac{r}{c}\right) = -\theta$$

$$\Rightarrow \quad \frac{r}{c} = e^{-\theta}$$

$$\Rightarrow \quad r = ce^{-\theta}$$

∴ $r = ce^{-\theta}$ is the required orthogonal trajectory.

Q42. Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$, λ being a parameter.

Answer : Dec.-13, Q11(b)

Given,

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda} = 1 \quad \text{... (1)}$$

Differentiating with respect to x,

$$\frac{1}{a^2+\lambda}(2x) + \frac{1}{b^2+\lambda}(2y)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{y}{b^2+\lambda}\frac{dy}{dx} = \frac{-x}{a^2+\lambda}$$

$$\Rightarrow (a^2+\lambda)y\frac{dy}{dx} = -x(b^2+\lambda)$$

Let, $\frac{dy}{dx} = y_1$

$\Rightarrow \quad (a^2 + \lambda)yy_1 = -x(b^2 + \lambda)$

$\Rightarrow \quad a^2yy_1 + \lambda yy_1 = -xb^2 - x\lambda$

$\Rightarrow \quad \lambda yy_1 + \lambda x = -b^2x - a^2yy_1$

$\Rightarrow \quad \lambda(x + yy_1) = -b^2x - a^2yy_1$

$\therefore \quad \lambda = -\frac{(b^2x + a^2yy_1)}{x + yy_1}$

Consider,

$$a^2 + \lambda = a^2 - \frac{(b^2x + a^2yy_1)}{x + yy_1}$$

$$\Rightarrow \quad a^2 + \lambda = \frac{a^2x + a^2yy_1 - b^2x - a^2yy_1}{x + yy_1}$$

$$\therefore \quad a^2 + \lambda = \frac{(a^2 - b^2)x}{x + yy_1} \qquad \text{... (2)}$$

Consider,

$$b^2 + \lambda = b^2 - \frac{(b^2x + a^2yy_1)}{x + yy_1}$$

$$= \frac{b^2x + b^2yy_1 - a^2yy_1 - b^2x}{x + yy_1}$$

$$\therefore \quad b^2 + \lambda = -\frac{(a^2 - b^2)yy_1}{x + yy_1} \qquad \text{... (3)}$$

Substituting equation (2) and equation (3) equation (1),

$$\frac{x^2}{\left[\frac{(a^2 - b^2)x}{x + yy_1}\right]} + \frac{y^2}{\left[\frac{-(a^2 - b^2)yy_1}{x + yy_1}\right]} = 1$$

$$\Rightarrow \quad \frac{x^2(x + yy_1)}{(a^2 - b^2)x} - \frac{y^2(x + yy_1)}{(a^2 - b^2)yy_1} = 1$$

$$\Rightarrow \quad (x^2 + x\,yy_1) - \frac{(xy + y^2y_1)}{y_1} = a^2 - b^2$$

$$\Rightarrow \quad y_1x^2 + x\,yy_1^2 - xy - y^2y_1 = y_1(a^2 - b^2)$$

$$\therefore \quad (x + yy_1)(xy_1 - y) = y_1(a^2 - b^2), \qquad \text{... (3)}$$

$\therefore$ Equation (3) is the Differential Equation (DE) of the given family.

For orthogonal trajectory replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in equation (3),

$$\left[x-\frac{y}{y_1}\right]\left[-\frac{x}{y_1}-y\right]=-\frac{1}{y_1}\left[a^2-b^2\right]$$

$$\Rightarrow \quad \left[\frac{xy_1-y}{y_1}\right]\left[\frac{-x-yy_1}{y_1}\right]=\frac{-(a^2-b^2)}{y_1}$$

$$\therefore \quad [xy_1-y]\,[x+yy_1]=y_1(a^2-b^2) \qquad \text{... (4)}$$

Equation (4) and equation (3) are same, i.e., D.E of the given family and equation of the O.T are same. Therefore, the given family of curves $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$ is self orthogonal.

Q43. Show that the family of curves $\frac{x^2}{c}+\frac{y^2}{c+2}+1=0$ is self orthogonal.

Answer : **(Model Paper-1, Q16(a) | June-11, Q11(b))**

Given that,

$$\frac{x^2}{c}+\frac{y^2}{c+2}+1=0 \qquad \text{... (1)}$$

Differentiating the above equation with respect to 'x',

$$\frac{2x}{c}+\frac{2y}{c+2}\frac{dy}{dx}=0$$

$$\Rightarrow \quad 2\left[\frac{x}{c}+\frac{y}{c+2}\frac{dy}{dx}\right]=0$$

$$\Rightarrow \quad \frac{x}{c}+\frac{y}{c+2}\frac{dy}{dx}=0$$

Let, $\frac{dy}{dx}=y_1$

$$\therefore \quad \frac{x}{c}+\frac{y}{c+2}y_1=0$$

$$\Rightarrow \quad x(c+2)+cyy_1=0$$

$$\Rightarrow \quad xc+2x+cyy_1=0$$

$$\Rightarrow \quad c[x+yy_1]=-2x$$

$$\therefore \quad c=\frac{-2x}{x+yy_1} \qquad \text{... (2)}$$

Substituting equation (2) in equation (1),

$$\frac{x^2}{\frac{-2x}{x+yy_1}}+\frac{y^2}{\left(\frac{-2x}{x+yy_1}\right)+2}+1=0$$

$$\Rightarrow \quad \frac{x^2(x+yy_1)}{-2x}+\frac{y^2(x+yy_1)}{-2x+2(x+yy_1)}+1=0$$

$$\Rightarrow \quad \frac{x^2(x+yy_1)}{-2x}+\frac{y^2(x+yy_1)}{-2x+2x+2yy_1}+1=0$$

$$\Rightarrow \quad \frac{x^2(x+yy_1)}{-2x} + \frac{y^2(x+yy_1)}{2yy_1} + 1 = 0$$

$$\Rightarrow \quad \frac{x(x+yy_1)}{2} - \frac{y(x+yy_1)}{2y_1} = 1$$

$$\Rightarrow \quad \frac{2xy_1(x+yy_1)-2y(x+yy_1)}{4y_1} = 1$$

$$\Rightarrow \quad (x+yy_1)(2xy_1-2y) = 4y_1$$

$$\Rightarrow \quad (x+yy_1)(xy_1-y) = 2y_1$$

$$\Rightarrow \quad x^2y_1 - xy + xy\,y_1^2 - y^2y_1 = 2y_1 \qquad \text{... (3)}$$

Replacing y_1 by $= \frac{-1}{y_1}$ and y_1^2 by $\left(\frac{-1}{y_1}\right)^2$ we get the required orthogonal trajectories (O.T)

i.e., $x^2\left(\frac{-1}{y_1}\right) - xy + xy\left(\frac{-1}{y_1}\right)^2 - y^2\left(\frac{-1}{y_1}\right) = 2\left(\frac{-1}{y_1}\right)$

$$\Rightarrow \quad \frac{-x^2}{y_1} - xy + \frac{xy}{y_1^2} + \frac{y^2}{y_1} = \frac{-2}{y_1}$$

$$\Rightarrow \quad \frac{-x^2y_1 - xyy_1^2 + xy + y^2y_1}{y_1^2} = \frac{-2}{y_1}$$

$$\Rightarrow \quad \frac{-1}{y_1}\left[\frac{x^2y_1 + xyy_1^2 - xy - y^2y_1}{y_1}\right] = \frac{-2}{y_1}$$

$$\Rightarrow \quad \frac{x^2y_1 + xyy_1^2 - xy - y^2y_1}{y_1} = 2$$

$$\therefore \quad x^2y_1 + xy\,y_1^2 - xy - y^2y_1 = 2y \qquad \text{... (4)}$$

Equations (3) and (4) are same i.e., D.E of the given family and equation of the O.T are same. Hence, the given equation is self orthogonal.

Q44. Show that the family of parabolas $x^2 = 4a(y + a)$ is self orthogonal.

Answer : April-16, Q11(a)

Given family of parabolas,

$$x^2 = 4a(y+a)$$

$$\Rightarrow \quad x^2 = 4ay + 4a^2 \qquad \text{... (1)}$$

Differentiating equation (1) with respect to 'y'

$$2x\frac{dx}{dy} = 4a + 0$$

$$\Rightarrow \quad \frac{2x}{4}\frac{dx}{dy} = a$$

$$\Rightarrow \quad \frac{x}{2}\frac{dx}{dy} = a \qquad \text{... (2)}$$

Substituting equation (2) in equation (1)

$$x^2 = 4\left[\frac{x}{2}\frac{dx}{dy}\right]^2 + 4y\left[\frac{x}{2}\frac{dx}{dy}\right]$$

$$\Rightarrow \quad x^2 = x^2\left(\frac{dx}{dy}\right)^2 + 2xy\left(\frac{dx}{dy}\right) \qquad \text{... (3)}$$

Replacing, $\frac{dx}{dy}$ by $\frac{-dy}{dx}$ we get the required orthogonal trajectory,

i.e., $$x^2 = x^2\left(\frac{-dy}{dx}\right)^2 + 2xy\left(\frac{-dy}{dx}\right)$$

$$\Rightarrow \quad x^2 = x^2\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right)$$

Multiplying both sides by $\left(\frac{dx}{dy}\right)^2$

$$x^2\left(\frac{dx}{dy}\right)^2 = x^2\left(\frac{dy}{dx}\right)^2\left(\frac{dx}{dy}\right)^2 - 2xy\left(\frac{dy}{dx}\right)\left(\frac{dx}{dy}\right)^2$$

$$\Rightarrow \quad x^2\left(\frac{dx}{dy}\right)^2 = x^2 - 2xy\left(\frac{dx}{dy}\right)$$

$$\Rightarrow \quad x^2\left(\frac{dx}{dy}\right)^2 + 2xy\,\frac{dx}{dy} = x^2$$

$$\Rightarrow \quad x^2 = x^2\left(\frac{dx}{dy}\right)^2 + 2xy\left(\frac{dx}{dy}\right) \qquad \text{... (4)}$$

Equations (3) and (4) are same.

$\therefore$ The given equation is self orthogonal.

UNIT 3

DIFFERENTIAL EQUATIONS OF HIGHER ORDERS

PART-A
SHORT QUESTIONS WITH SOLUTIONS

Q1. Solve y" – 5y' + 6y = 0.

Answer :

Given differential equation is,

$y'' - 5y' + 6y = 0$

$(D^2 - 5D + 6)y = 0$

Let,

$f(D) = D^2 - 5D + 6$

The auxiliary equation is $f(m) = 0$

$m^2 - 5m + 6 = 0$

$\Rightarrow \quad m^2 - 3m - 2m + 6 = 0$

$\Rightarrow \quad (m - 3)(m - 2) = 0$

$\Rightarrow \quad m = 2, 3$

The roots are real and distinct

$\therefore$ The general solution is $y = c_1 e^{2x} + c_2 e^{3x}$

Q2. State the nth order homogeneous differential equation of type f(D)y = 0.

Answer :

A linear differential equation of the form $\frac{a_0 d^n y}{dx^n} + \frac{a_1 d^{n-1} y}{dx^{n-1}} + ... + a_n y = 0$, is known as n^{th} order homogeneous differential equation.

The symbolic form of above equation is,

$(a_0 D^n + a_1 D^{n-1} + ... a_n)y = 0$

or

$f(D)y = 0$

Where,

$f(D)$ polynomial of n^{th} order.

Q3. Solve (D^4 + D^2 + 1) y = 0.

Answer : (Model Paper-1, Q5 | May/June-12, Q3)

Given differential equation is,

$(D^4 + D^2 + 1)y = 0$

Auxiliary equation is,

$$f(m) = 0$$

i.e., $m^4 + m^2 + 1 = 0$

$\Rightarrow$ $m^4 + 2m^2 + 1 - m^2 = 0$

$\Rightarrow$ $(m^2 + 1)^2 - m^2 = 0$

$\Rightarrow$ $(m^2 + m + 1)(m^2 - m + 1) = 0$

$\Rightarrow$ $m^2 + m + 1 = 0$ or $m^2 - m + 1 = 0$

$$m = \frac{-1 \pm \sqrt{1-4}}{2} \text{ or } m = \frac{1 \pm \sqrt{1-4}}{2}$$

$$m = \frac{-1 \pm \sqrt{-3}}{2} \text{ or } m = \frac{1 \pm \sqrt{-3}}{2}$$

$$m = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \text{ or } m = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$\therefore$ The roots are complex, hence the general solution is,

$$y = e^{-\frac{x}{2}}\left[C_1 \cos\frac{\sqrt{3}}{2}x + C_2 \sin\frac{\sqrt{3}}{2}x\right] + e^{\frac{x}{2}}\left[C_3 \cos\frac{\sqrt{3}}{2}x + C_4 \sin\frac{\sqrt{3}}{2}x\right]$$

Q4. Solve y" – y = 0, when y = 0 and y' = 2 at x = 0.

June-14, Q3

OR

Solve y" – y = 0, y(0) = 0, y'(0) = 2.

Answer : **(June-13, Q3 | June-11, Q4 | May/June-09, Q4)**

Given differential equation is,

$$y'' - y = 0 \quad \text{... (1)}$$

Initial conditions are,

$$y(0) = 0; y'(0) = 2$$

Equation (1) in symbolic form is represented as,

$$D^2y - y = 0 \Rightarrow (D^2 - 1)y = 0$$

Let,

$$f(D) = D^2 - 1$$

The auxiliary equation is $f(m) = 0$

$\Rightarrow$ $m^2 - 1 = 0$ ... (2)

$\Rightarrow$ $(m + 1)(m - 1) = 0$

$\Rightarrow$ $m = 1, m = -1$

$\therefore$ The roots of equation (2) are $m = 1, -1$.

Since the roots are real and distinct, the general solution of equation (1) is,

$$y = C_1e^{-x} + C_2 e^{1.x}$$

$$\therefore \quad y = C_1e^{-x} + C_2e^{x}$$

Where, C_1 and C_2 are constants

$\Rightarrow$ $y(x) = C_1e^{-x} + C_2e^{x}$ ... (3)

Substituting $x = 0$ in the above equation,

$$y(0) = C_1e^{-0} + C_2 e^{0}$$

$\Rightarrow$ $y(0) = C_1 + C_2$

But, $y(0) = 0$

$\therefore C_1 + C_2 = 0$ ... (4)

Differentiating equation (3) with respect to 'x',

$y'(x) = -C_1e^{-x} + C_2.e^x$

$y'(x) = C_2e^x - C_1.e^{-x}$

Substituting $x = 0$ in the above equation,

$y'(0) = C_2\, e^0 - C_1.e^{-0}$

$y'(0) = C_2 - C_1$

But, $y'(0) = 2$

$2 = C_2 - C_1$

$\Rightarrow \quad C_1 = C_2 - 2$ ... (5)

Substituting equation (5) in equation (4),

$C_2 - 2 + C_2 = 0 \Rightarrow C_2 = 1$

$\therefore C_1 = -1$ and $C_2 = 1$

Substituting the values of C_1 and C_2 in equation (3),

$\therefore \quad y(x) = e^x - e^{-x}$

Q5. Solve y'' + y' – 2y = 0, y(0) = 0, y'(0) = 3.

Answer : **(Model Paper-2, Q5 | Jan.-12, Q4)**

Given differential equation is,

$y'' + y' - 2y = 0$

$\Rightarrow \quad \frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$ ... (1)

Initial conditions are,

$y(0) = 0,$

$y'(0) = 3$

Equation (1) in symbolic form is represented as,

$D^2y + Dy - 2y = 0$

$\Rightarrow \quad (D^2 + D - 2)y = 0$

Auxiliary equation is $f(m) = 0$

i.e., $m^2 + m - 2 = 0$

$\Rightarrow \quad m^2 + 2m - m - 2 = 0$

$\Rightarrow \quad (m + 2)(m - 1) = 0$

$\Rightarrow \quad m = 1, -2$

$\therefore$ The roots of the equation (1) are real and distinct,

Hence, the general solution of equation (1) is given as,

$y(x) = c_1e^x + c_2e^{-2x}$ ... (2)

Substituting $x = 0$ in equation (2),

$y(0) = c_1e^{(0)} + c_2e^{-2(0)}$

$\Rightarrow \quad y(0) = c_1e^0 + c_2e^0$

$\Rightarrow \quad y(0) = c_1(1) + c_2(1)$

$\Rightarrow \quad 0 = c_1 + c_2 \quad [\because y(0) = 0]$ ... (3)

Differentiating equation (2), with respect to 'x',

$\Rightarrow \quad y'(x) = \frac{d}{dx}[c_1e^x + c_2e^{-2x}]$

$\Rightarrow \quad y'(x) = c_1\frac{d}{dx}[e^x] + c_2\frac{d}{dx}[e^{-2x}]$

$\Rightarrow \quad y'(x) = c_1[e^x(1)] + c_2[e^{-2x}(-2)]$

$\Rightarrow \quad y'(x) = c_1e^x - 2c_2e^{-2x}$ ... (4)

Substituting $x = 0$, in equation (4),

$y'(0) = c_1.e^{(0)} - 2c_2e^{-2(0)}$

$\Rightarrow \quad y'(0) = c.e^0 - 2c_2e^0$

$\Rightarrow \quad y'(0) = c_1(1) - 2c_2(1)$

$\Rightarrow \quad y'(0) = c_1 - 2c_2$

$\Rightarrow \quad c_1 - 2c_2 = 3 \quad [\because y'(0) = 3]$ ... (5)

Solving equations (3) and (5),

$\Rightarrow c_2 = -1$

Substituting c_2 value in equation (2),

$c_1 + c_2 = 0$

$\Rightarrow \quad c_1 - 1 = 0$

$\Rightarrow \quad c_1 = 1$

Substituting c_1, c_2 values in equation (2),

$y(x) = (1)e^{-x} + (-1)e^{-2x} = e^{-x} - e^{-2x}$

$\therefore y(x) = e^{-x} - e^{-2x}$

Q6. If $y_1 = e^x$ is one of the solutions of y'' + 3y' – 4y = 0, then find general solution, by reducing order of differential equation.

Answer : **Jan.-12, Q13(b)**

Given differential equation is,

$y'' + 3y' - 4y = 0$

$\Rightarrow \quad (D^2 + 3D - 4)y = 0$

$\therefore \quad f(D) = D^2 + 3D - 4$

The auxilliary equation is $f(m) = 0$

$\Rightarrow \quad m^2 + 3m - 4 = 0$

$\Rightarrow \quad m^2 + 4m - 1m - 4 = 0$

$\Rightarrow \quad (m + 4)(m - 1) = 0$

$\Rightarrow \quad m = -4, 1$

$\therefore$ The general solution is $y = c_1e^{1x} + c_2e^{-4x}$

$\Rightarrow y = c_1e^x + c_2e^{-4x}$

Given that $y_1 = e^x$ is one solution, then the second linearly independent solution is $y_2 = e^{-4x}$.

$\therefore y_2 = e^{-4x}$

Q7. Find the particular integral of (D^2 + 1) y = 8 e^{-x}.

Answer : June-14, Q4

Given differential equation is,

$$(D^2 + 1)y = 8e^{-x} \quad \text{... (1)}$$

Equation (1) is a non-homogeneous linear differential equation of the form,

$$f(D)y = X \quad \text{... (2)}$$

Comparing equation (2) with equation (1),

$$f(D) = D^2 + 1 \text{ and } X = 8e^{-x}$$

The particular integral $(P.I) = \dfrac{X}{f(D)}$

$$\Rightarrow \quad y_p = \frac{8e^{-x}}{D^2+1}$$

$$= \frac{8e^{-x}}{(-1)^2+1}$$

$$= \frac{8e^{-x}}{2} = 4e^{-x} \quad [\because D = -1]$$

$\therefore$ Particular integral is $4e^{-x}$

Q8. Find the particular integral of $\dfrac{d^2y}{dx^2}$ + y = cosh 3x

Answer : Dec.-13, Q3

Given differential equation is,

$$\frac{d^2y}{dx^2} + y = \cosh 3x$$

Equation in symbolic form is given as,

$$(D^2 + 1)y = \cosh 3x \quad \text{... (1)}$$

Particular integral is given as,

$$P.I = y_p = \frac{\cosh 3x}{D^2+1}$$

$$= \frac{e^{3x}+e^{-3x}}{2(D^2+1)} \quad \left[\because \cosh x = \frac{e^x+e^{-x}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{e^{3x}}{D^2+1} + \frac{e^{-3x}}{D^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{e^{3x}}{3^2+1} + \frac{e^{-3x}}{(-3)^2+1}\right]$$

$$[\because D = 3 \text{ and } -3]$$

$$= \frac{1}{2}\left[\frac{e^{3x}}{10} + \frac{e^{-3x}}{10}\right]$$

$$= \frac{1}{2}\left[\frac{e^{3x}+e^{-3x}}{10}\right]$$

$$= \frac{1}{20}[e^{3x}+e^{-3x}]$$

$$\therefore y_p = \frac{1}{20}[e^{3x}+e^{-3x}]$$

Q9. Find the particular integral of (D^2 – 1)y = 8e^{3x}.

Answer : June-13, Q4

Given differential equation is,

$$(D^2 - 1)\, y = 8e^{3x} \quad \text{... (1)}$$

Equation (1) is a non-homogeneous linear differential equation of the form,

$$f(D)\, y = X \quad \text{... (2)}$$

Comparing equation (2) with equation (1),

$$f(D) = D^2 - 1,\ X = 8\, e^{3x}$$

$$f(D) = D^2 - 1 = 0$$

$$\Rightarrow \quad D^2 = 1$$

$$\Rightarrow \quad D = \pm 1$$

The particular integral of the given differential equation is,

$$y_p = \frac{X}{f(D)}$$

$$\Rightarrow \quad y_p = \frac{8e^{3x}}{(D^2-1)} = \frac{8e^{3x}}{(3)^2-1}$$

$$= \frac{8e^{3x}}{8} = e^{3x}$$

$$\therefore y_p = e^{3x}$$

Q10. Find the particular integral of (D^2 + 1) y = cos x.

Answer : Dec.-12, Q4

Given differential equation is,

$$(D^2 + 1)y = \cos x \quad \text{... (1)}$$

Particular integral is given as,

$$P.I = y_p = \frac{\cos x}{D^2+1}$$

$$= \frac{x\sin(1)x}{2(1)} \quad \left[\begin{array}{l}\because D^2 = -1 \Rightarrow f(D) = 0 \\ \Rightarrow \dfrac{\cos bx}{D^2+b^2} = \dfrac{x\sin bx}{2b}\end{array}\right]$$

$$= \frac{x\sin x}{2}$$

$$\therefore y_p = \frac{x\sin x}{2}$$

Q11. Solve (D^2 + 4D + 5)y = 2e^{-2x}.

Answer : (Model Paper-3, Q5 | Dec.-13, Q4)

Given differential equation is,

$$(D^2 + 4D + 5)y = 2e^{-2x}$$

Auxiliary equation is $m^2 + 4m + 5 = 0$

$$m = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2}$$

$$= -2 \pm i$$

$$\therefore \quad m = -2 - i, -2 + i$$

The complimentary function $C.F = c_1 e^{-2x}\cos x + c_2 e^{-2x}\sin x$

$= e^{-2x}(c_1 \cos x + c_2 \sin x)$

$\therefore y_c = e^{-2x}(c_1 \cos x + c_2 \sin x)$

The particular integral (P.I) is,

$$y_p = \frac{2e^{-2x}}{D^2+4D+5}$$

$$= \frac{2e^{-2x}}{(-2)^2+4(-2)+5} = 2e^{-2x} \quad [\because D = -2]$$

$\therefore y_p = 2e^{-2x}$

The general solution is given as,

$$y = y_c + y_p$$

$$= e^{-2x}(c_1\cos x + c_2\sin x) + 2e^{-2x}$$

$$= e^{-2x}(c_1\cos x + c_2\sin x + 2)$$

$$\therefore \quad y = e^{-2x}(c_1\cos x + c_2\sin x + 2)$$

Q12. Solve (D^2 + 9) y = sin 3x.

Answer : **May/June-12, Q4**

Given differential equation is,

$(D^2 + 9)y = \sin 3x$

Auxiliary equation is $f(m) = 0$

i.e., $m^2 + 9 = 0 \Rightarrow m^2 = -9$

$m = \pm 3i$

$\therefore$ The roots are complex conjugate.

Hence, complementary function is given as,

$y_c = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$

$\Rightarrow \quad y_c = e^{0.x}(c_1 \cos 3x + c_2 \sin 3x)$

(Here $\alpha = 0, \beta = 3$)

$\therefore y_c = c_1 \cos 3x + c_2 \sin 3x$

The particular integral is given as,

$$y_p = \frac{1}{D^2+9}\sin 3x$$

$$\Rightarrow \quad y_p = \frac{1}{D^2+3^2}\sin 3x$$

$$= -\frac{x}{2}\left(\frac{1}{3}\cos 3x\right) \quad \left[\begin{array}{l}\because D^2 = -9 \to f(-3)^2 \\ \Rightarrow \dfrac{\sin ax}{D^2+a^2} = -\dfrac{x\cos ax}{2a}\end{array}\right]$$

$$y_p = \frac{-x}{6}\cos 3x$$

The general solution of given differential equation is,

$y = y_c + y_p$

$\therefore \ y = c_1 \cos 3x + c_2 \sin 3x - \dfrac{x}{6}\cos 3x.$

Q13. Find a particular integral of $\dfrac{d^3y}{dx^3} - y = (e^x + 1)^2$.

Answer : **Model Paper-3, Q6**

Given differential equation is,

$$\frac{d^3y}{dx^3} - y = (e^x + 1)^2$$

The given equation can be written in symbolic form as,

$$(D^3 - 1)y = (e^x + 1)^2 \quad \ldots(1)$$

Equation (1) is a non - homogeneous linear differential equation of the form,

$$f(D)y = X \quad \ldots(2)$$

Comparing equation (2) with equation (1),

$f(D) = D^3 - 1$ and $X = (e^x + 1)^2$

The auxiliary equation is $(m^3 - 1) = 0$

$\Rightarrow \quad (m - 1)(m^2 + m + 1) = 0$

$\Rightarrow \quad (m - 1) = 0, m^2 + m + 1 = 0$

$$m = 1,\ m = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm i\sqrt{3}}{2}$$

$$= -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

The complementary function is,

$$C.F = y_c = C_1 e^x + e^{-\frac{x}{2}}\left[C_2\cos\left(\frac{\sqrt{3}x}{2}\right) + C_3\sin\left(\frac{\sqrt{3}x}{2}\right)\right]$$

Particular integral,

$$y_p = \frac{1}{f(D)}.X$$

$$y_p = \frac{1}{(D^3-1)}.(e^x+1)^2$$

$$= \frac{e^{2x}+2e^x+1}{(D^3-1)}$$

$$= \frac{1}{(D^3-1)}e^{2x} + 2\frac{1}{(D^3-1)}e^x + \frac{1}{(D^3-1)}.e^{0x}$$

$$= \frac{1}{(2^3-1)}e^{2x} + 2\frac{1}{3D^2}e^x + \frac{1}{(0-1)}$$

$$= \frac{e^{2x}}{7} + \frac{2}{3}e^x - 1$$

The complete solution is $y = y_c + y_p$

$$y = C_1e^x + e^{-\frac{x}{2}}\left[C_2\cos\left(\frac{\sqrt{3}x}{2}\right) + C_3\sin\left(\frac{\sqrt{3}x}{2}\right)\right] + \frac{e^{2x}}{7} + \frac{2}{3}e^x - 1$$

Q14. Solve by the method of variation of parameters y" + y = sec x.

Answer : Model Paper-1, Q6

Given differential equation is,

$$y'' + y = \sec x$$

$$\Rightarrow \frac{d^2y}{dx^2} + y = \sec x$$

$$\Rightarrow (D^2 + 1)\,y = \sec x \quad \text{... (1)}$$

Equation (1) is a non homogeneous linear differential equation of the form,

$$f(D)y = Q(x)$$

Comparing equations (1) and (2), ... (2)

$$f(D) = D^2 + 1,\ Q(x) = \sec x$$

General solution is given as,

$$y = y_c + y_p \quad \text{... (3)}$$

Auxiliary equation is, $f(m) = 0$

$$\Rightarrow m^2 + 1 = 0$$

$$\Rightarrow m = \pm i$$

Roots are complex conjugate of the form $\alpha \pm i\beta$

The complementary function is given as,

$$y_c = c_1\cos x + c_2\sin x \quad \text{... (4)}$$

The particular integral (P.I) is,

$$y_p = Pf_1 + Rf_2 \quad \text{... (5)}$$

Where,

$$f_1 = \cos x, f_2 = \sin x$$

$$f_1' = -\sin x,\ f_2' = \cos x$$

$$f_1 f_2' - f_1' f_2 = \sin^2 x + \cos^2 x$$

$$= 1$$

$$P = -\int \frac{f_2 Q(x)}{f_1 f_2' - f_1' f_2}\,dx$$

$$= -\int \frac{\sin x.\sec x}{1}\,dx$$

$$= -\int \frac{\sin x}{\cos x}\,dx$$

$$= -\int \tan x\,dx$$

$$= -(-\log(\cos x)) = \log(\cos x)$$

$$R = \int \frac{f_1 Q(x)}{f_1 f_2' - f_1' f_2}.dx$$

$$= \int \frac{\cos x.\sec x}{1}\,dx$$

$$= \int 1\,dx$$

$$= x$$

Substituting the corresponding values in equation (5),

$$y_p = \log(\cos x)\cos x + x.\sin x$$

$$\Rightarrow y_p = \cos x.\log(\cos x) + x\sin x$$

Substituting the corresponding values in equation (3)

$$y = c_1\cos x + c_2\sin x + \cos x\log(\cos x) + x\sin x.$$

Q15. Using the method of variation of parameters solve (D² + 1) y = x.

Answer : [Model Paper-2, Q6 | June-14, Q12(a)]

Given differential equation is,

$$(D^2 + 1)y = x$$

Let, $f(D) = D^2 + 1$, $Q(x) = x$

The auxiliary equation is,

$$f(m) = 0$$

$$\Rightarrow m^2 + 1 = 0$$

$$\Rightarrow m^2 = -1$$

$$\Rightarrow m = \pm i$$

$$\Rightarrow m = i, -i$$

The complimentary function is given as,

$$y_c = c_1\cos x + c_2\sin x$$

$$\therefore y_c = c_1\cos x + c_2\sin x$$

The particular integral is given as,

$$y_p = Pf_1 + Rf_2 \quad \text{... (1)}$$

Where,

$$f_1 = \cos x, f_2 = \sin x$$

$$f_1' = -\sin x,\ f_2' = \cos x$$

$$f_1 f_2' - f_2 f_1' = \cos x(\cos x) - (\sin x)(\sin x)$$

$$= \cos^2 x + \sin^2 x$$

$$= 1$$

$$P = -\int \frac{f_2 Q(x)}{f_1 f_2' - f_2 f_1'}\,dx$$

$$= -\int x\sin x.dx$$

$$= -\left[x(-\cos x) - \int -\cos x.dx\right]$$

$$= -[-x\cos x + \sin x]$$

$$= x\cos x - \sin x$$

$$R = \int \frac{f_1 Q(x)}{f_1 f_2' - f_2 f_1'}\,dx$$

$$= \int \frac{x\cos x}{1}\,dx$$

$$= \int x\cos x\,dx$$

$$= x\sin x - \int 1.\sin x\,dx$$

$$= x\sin x - (-\cos x)$$

$$= x\sin x + \cos x$$

Substituting the corresponding values in equation (1),

$y_p = [x\cos x - \sin x]\cos x + [x\sin x + \cos x]\sin x$

$= x\cos^2 x - \sin x\cos x + x\sin^2 x + \cos x\sin x$

$= x(\sin^2 x + \cos^2 x)$

$\therefore\ y_p = x$

The general solution is given as,

$y = y_c + y_p$

$\therefore\quad y = c_1\cos x + c_2\sin x + x.$

Q16. Find the particular integral of

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 9y = 10 + \frac{5}{x^2}.$$

Answer :

Given differential equation is,

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 9y = 10 + \frac{5}{x^2}$$

$\Rightarrow\quad (x^2D^2 + xD - 9)y = 10 + \frac{5}{x^2}$ ... (1)

Let,

$x = e^z \Rightarrow z = \log x$

$\Rightarrow\quad xD \equiv D'$

$x^2D^2 \equiv D'(D'-1),\ D' \equiv \frac{d}{dz}$ ... (2)

Substituting equation (2) in equation (1)

$(D'(D'-1) + D' - 9)y = 10 + 5e^{-2z}$

$\Rightarrow\quad [(D')^2 - 9]y = 10 + 5e^{-2z}$

The particular integral is given as,

$$\text{P.I} = \frac{10 + 5e^{-2z}}{(D')^2 - 9}$$

$$= \frac{10e^{0.z}}{(D')^2 - 9} + \frac{5e^{-2z}}{(D')^2 - 9}$$

$$= \frac{10}{0-9} + \frac{5e^{-2z}}{(-2)^2 - 9}$$

$$= -\frac{10}{9} - e^{-2z}$$

$\therefore\quad$ Particular integral is, $-\frac{10}{9} - e^{-2z}$

Where, $z = \log x$.

Q17. Find the particular integral of

$$(x+2)^2\frac{d^2y}{dx^2} - (x+2)\frac{dy}{dx} + y = 3x + 4.$$

Answer :

Given differential equation is,

$$(x+2)^2\frac{d^2y}{dx^2} - (x+2)\frac{dy}{dx} + y = 3x + 4$$

$[(x+2)^2D^2 - (x+2)D + 1]y = 3x + 4$ ... (1)

Let,

$x + 2 = e^z \Rightarrow z = \log(x+2)$

$\Rightarrow\ (x+2)D \equiv D',\ (x+2)^2 \equiv D'(D'-1),\ D' \equiv \frac{d}{dz}$ ... (2)

Substituting equation (2) in equation (1),

$[D'(D'-1) - D' + 1]y = 3[e^z - 2] + 4$

$\Rightarrow\quad [(D')^2 - 2D' + 1]y = 3e^z - 2$

The particular integral is given as,

$$\text{P.I} = \frac{3e^z - 2}{(D')^2 - 2D' + 1}$$

$$= \frac{3e^z}{(D')^2 - 2D' + 1} - \frac{2e^{0.z}}{(D')^2 - 2D' + 1}$$

$$= \frac{3e^z}{1-2+1} - \frac{2}{0-0+1}$$

$$= \frac{3e^z}{0} - \frac{2}{1}$$

$$= \frac{3z.e^z}{2D' - 2} - 2$$

$$= \frac{3ze^z}{0} - 2 = \frac{3z^2e^z}{2} - 2$$

$\therefore\quad$ The particular integral is, $\frac{3z^2e^z}{2} - 2$

Where, $z = \log(x+2)$.

PART-B

ESSAY QUESTIONS WITH SOLUTIONS

3.1 SOLUTIONS OF SECOND AND HIGHER ORDER LINEAR HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

Q18. Write about homogeneous second order differential equations of the type f(D)y = 0.

Answer :

A linear differential equation is of the form,

$$a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0 \qquad \text{... (1)}$$

Where, right hand side (R.H.S) of the equation is zero and is known as a homogeneous second order differential equation.

Where,

$a_0\ a_1$ and a_2 – Constant coefficients

If the terms $\frac{d^2}{dx^2}$, $\frac{d}{dx}$ are replaced with D^2 and D respectively, then equation (1) becomes,

$$a_0 D^2y + a_1 Dy + a_2 y = 0$$

$$\Rightarrow \quad (a_0 D^2 + a_1 D + a_2)y = 0 \qquad \text{... (2)}$$

Equation (2) is the symbolic (operator) form of equation (1), and is of the type $f(D)y = 0$.

Therefore, $f(D)y = 0$ is the standard form of second order homogeneous linear differential equations.

The steps involved in solving a second order homogeneous differential equation of type $f(D)y = 0$ are,

(i) In the first step, write the equation $a_0 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 = 0$ in the symbolic form, $(a_0 D^2 + a_1 D + a_2)y = 0$

(ii) In the next step, determine the auxiliary equation (A.E), taking $f(D) = 0$ i.e., $(a_0 D^2 + a_1 D + a_2) = 0$

(iii) Determine the roots of auxiliary equation and write its corresponding general solutions (or) complementary functions.

Depending on the nature of roots, different forms of general solutions are shown in table below.

S.No.	Nature of Roots	General Solution
1.	Real and equal roots (m, m)	$y = (c_1 + c_2x)\, e^{mx}$
2.	Real and distinct roots (m_1, m_2)	$y = c_1e^{m_1x} + c_2e^{m_2x}$
3.	Complex conjugate roots $(\alpha \pm i\beta)$	$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$
4.	A pair of surd roots $(\alpha \pm \sqrt{\beta})$	$y = e^{\alpha x}(c_1 \cosh\sqrt{\beta}\,x + c_2 \sinh\sqrt{\beta}\,x)$

Table

Q19. Solve $(D^2 - 6D + 18)y = 0$

Answer :

Given differential equation is,

$$(D^2 - 6D + 18)y = 0$$

Auxiliary equation is $f(m) = 0$

$$m^2 - 6m + 18 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \times 1 \times 18}}{2 \times 1}$$

$$= \frac{6 \pm \sqrt{-36}}{2}$$

$$m = \frac{6 \pm 6i}{2}$$

$\Rightarrow$ $m = 3 \pm 3i$

$m = 3 + 3i, 3 - 3i$

The roots are complex conjugate of the form $\alpha \pm i\beta$

The general solution is given as,

$$y(x) = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

$\therefore$ $y = e^{3x}(c_1 \cos 3x + c_2 \sin 3x)$.

Q20. Solve the initial value problem y'' – 2y' + 3y = 0 with y(0) = 1, y'(0) = 0.

Answer : **(Model Paper-1, Q13(a) | June-14, Q16(b))**

Given differential equation is,

$y'' - 2y' + 3y = 0$ ... (1)

Initial points, $y(0) = 1, y'(0) = 0$

Equation (1) can be represented in symbolic form as,

$f(D) = D^2y - 2Dy + 3y = 0$

$\Rightarrow$ $(D^2 - 2D + 3)\, y = 0$

Auxiliary equation is, $f(m) = 0$ i.e., $m^2 - 2m + 3 = 0$

The roots of above equation are,

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{-8}}{2}$$

$$= \frac{2 \pm \sqrt{-4 \times 2}}{2}$$

$$= \frac{2 \pm 2i\sqrt{2}}{2}$$

$m = 1 \pm i\sqrt{2}$

$\Rightarrow$ $m = 1 + i\sqrt{2}, 1 - i\sqrt{2}$

$\therefore$ The roots are complex conjugate of the form $\alpha \pm i\beta$

Hence, the general solution of equation (1) is given as,

$$y(x) = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

$$\Rightarrow \quad y(x) = e^{x}(c_1 \cos \sqrt{2}\, x + c_2 \sin \sqrt{2}\, x) \qquad \text{... (2)}$$

Substituting the initial condition $y(0) = 1$ in equation (2),

$$y(0) = e^{(0)}[c_1 \cos \sqrt{2}(0) + c_2 \sin \sqrt{2}(0)]$$

$$1 = 1[c_1(1) + c_2(0)]$$

$$\therefore\ c_1 = 1$$

Differentiating equation (2) with respect to 'x',

$$y'(x) = \frac{d}{dx}[e^x(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x)]$$

$$= e^x\left[\frac{d}{dx}(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x)\right] + \left[c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x\right]\frac{d}{dx}e^x$$

$$= e^x\left[c_1(-\sin \sqrt{2}x(\sqrt{2}) + c_2 \cos \sqrt{2}x(\sqrt{2}))\right] + \left[c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x\right]e^x$$

$$y'(x) = e^x\left[-\sqrt{2}c_1 \sin \sqrt{2}x + \sqrt{2}c_2 \cos \sqrt{2}x\right] + e^x\left[c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x\right]$$

Substituting $y'(0) = 0$ in the above equation,

$$y'(0) = e^0\left[-\sqrt{2}c_1 \sin \sqrt{2}(0) + \sqrt{2}c_2 \cos \sqrt{2}(0)\right] + e^0\left[c_1 \cos \sqrt{2}(0) + c_2 \sin \sqrt{2}(0)\right]$$

$$0 = 1\left[0 + \sqrt{2}c_2(1)\right] + 1\left[c_1(1) + 0\right]$$

$$0 = \sqrt{2}c_2 + c_1 \qquad (\because\ c_1 = 1)$$

$$\Rightarrow \sqrt{2}c_2 + 1 = 0 \Rightarrow \sqrt{2}c_2 = -1$$

$$\therefore\ c_2 = -\frac{1}{\sqrt{2}}$$

Substituting the values of c_1 and c_2 in equation (2),

$$y(x) = e^x\left(1.\cos \sqrt{2}x + \left(\frac{-1}{\sqrt{2}}\right)\sin \sqrt{2}x\right)$$

$$\therefore\ y(x) = e^x\left(\cos \sqrt{2}x - \frac{1}{\sqrt{2}}\sin \sqrt{2}x\right)$$

Q21. Solve the initial value problem y'''– 5y'' + 7y' – 3y = 0, y(0) = 1, y'(0) = 0, y''(0) = –5.

Answer : June-13, Q12(b)

Given differential equation is,

$$y''' - 5y'' + 7y' - 3y = 0 \qquad \text{... (1)}$$

Initial points, $y(0) = 1$, $y'(0) = 0$, $y''(0) = -5$

Equation (1) in symbolic form can be expressed as,

$(D^3 - 5D^2 + 7D - 3)y = 0$

$\therefore \quad f(D) = D^3 - 5D^2 + 7D - 3$

Auxiliary equation is $f(m) = 0 \Rightarrow m^3 - 5m^2 + 7m - 3 = 0$

By trial and error method,

m = 1	1	– 5	7	– 3
	0	1	– 4	3
	1	– 4	3	0

$\Rightarrow \quad (m-1)(m^2 - 4m + 3) = 0$

$\Rightarrow \quad (m-1)(m^2 - 3m - m + 3) = 0$

$\Rightarrow \quad (m-1)(m(m-3) - 1(m-3)) = 0$

$\Rightarrow \quad (m-1)(m-1)(m-3) = 0$

$\Rightarrow \quad m = 1, 1, 3$

$\therefore$ Roots of auxiliary equations are, $m_1 = 1, 1; m_2 = 3$

The solution is given as,

$$y_c = (c_1 + c_2x)\, e^{m_1x} + c_3\, e^{m_2x} = (c_1 + c_2x)e^x + c_3e^{3x}$$

$\therefore \quad y(x) = (c_1 + c_2x)e^x + c_3e^{3x}$ ... (2)

Substituting $y(0) = 1$ i.e., $x = 0, y = 1$ in equation (2),

$\Rightarrow \quad y(0) = (c_1 + 0\, c_2)e^0 + c_3e^0$

$\Rightarrow \quad y(0) = c_1 + c_3$

$\Rightarrow \quad c_1 + c_3 = 1$ ... (3)

Differentiating equation (2) with respect to x,

$y'(x) = e^xc_2 + (c_1 + xc_2)e^x + c_3 3e^{3x}$ ... (4)

Substituting $y'(0) = 0$ i.e., $x = 0, y' = 0$ in equation (4),

$0 = e^0c_2 + (c_1 + 0.c_2)e^0 + c_3\, 3.e^0$

$0 = c_2 + c_1 + 3c_3$ ... (5)

Differentiating equation (4) with respect x,

$y''(x) = c_2e^x + c_2e^x + (c_1 + xc_2)e^x + c_3 9e^{3x}$ ...(6)

Substituting $y''(0) = -5$ i.e., $x = 0$ and $y'' = -5$ in equation (6),

$-5 = c_2e^0 + c_2e^0 + (c_1 + 0.c_2)e^0 + c_3 9e^{(0)}$

$-5 = c_2 + c_2 + (c_1) + 9c_3$

$-5 = c_1 + 2c_2 + 9c_3$ ... (7)

Solving equations (5) and (7),

$2 \times (c_1 + c_2 + 3c_3) = 0$

$$\begin{array}{l} 2c_1 + 2c_2 + 6c_3 = 0 \\ c_1 + 2c_2 + 9c_3 = -5 \\ -\quad -\quad -\quad + \\ \hline +c_1 - 3c_3 = 5 \end{array}$$ ... (8)

Solving equations (3) and (8),

$$\begin{array}{l} c_1 + c_3 = 1 \\ c_1 - 3c_3 = 5 \\ -\quad +\quad - \\ \hline 4c_3 = -4 \end{array}$$

$\therefore \quad c_3 = -1$

Substituting c_3 value in equation (8),

$c_1 - 3c_3 = 5$

$c_1 - 3(-1) = 5$

$\Rightarrow \quad c_1 = 2$

Substituting c_3 and c_1 values in equation (5),

$c_1 + c_2 + 3c_3 = 0$

$2 + c_2 + 3(-1) = 0$

$c_2 - 1 = 0$

$\Rightarrow \quad c_2 = 1$

$\therefore c_1 = 2, c_2 = 1, c_3 = -1$

The general solution is given as,

$y_c = (2 + x(1))e^x + (-1)e^{3x}$

$\therefore y_c = (2+x)e^x - e^{3x}$

Q22. Solve the initial value problem y''' – 2y'' – 5y' + 6y = 0, y(0) = 0, y'(0) = 0, y''(0) = 1.

Answer : **(Model Paper-2, Q13(a) | Dec.-12, Q12(b) | Jan.-12, Q12(b))**

Given differential equation is,

$y''' - 2y'' - 5y' + 6y = 0$

$y(0) = 0, y'(0) = 0, y''(0) = 1$

Symbolic form of given differential equation is,

$(D^3 - 2D^2 - 5D + 6)y = 0$

$f(D) = D^3 - 2D^2 - 5D + 6$

Auxiliary equation,

$f(m) = m^3 - 2m^2 - 5m + 6$

Substitute $m = 1$

$f(1) = 1 - 2 - 5 + 6 = -7 + 7 = 0$

$\therefore \quad m = 1$ is one root

By trial and error method,

m = 1	1	–2	–5	6
	0	1	–1	–6
	1	–1	–6	0

$\therefore \quad m^2 - m - 6 = 0$

$\Rightarrow \quad m^2 - 3m + 2m - 6 = 0$

$(m + 2)(m - 3) = 0$

$m = -2, 3$

$\therefore \quad m = 1, -2, 3$

$\therefore \quad y = c_1e^x + c_2e^{-2x} + c_3e^{3x}$ ... (1)

Differentiating equation (1) with respect to 'x',

$y' = c_1e^x(1) + c_2e^{-2x}(-2) + c_3e^{3x}(3)$

$y' = c_1e^x - 2c_2e^{-2x} + 3c_3e^{3x}$ ... (2)

Differentiating equation (2) with respect to 'x',

$y'' = c_1e^x(1) - 2c_2e^{-2x}(-2) + 3c_3e^{3x}(3)$

$y'' = c_1e^x + 4c_2e^{-2x} + 9c_3e^{3x}$ ... (3)

Given conditions are,

(i) $y(0) = 0$

$x = 0$ and $y = 0$

Substituting the corresponding values in equation (1),

$0 = c_1e^0 + c_2e^0 + c_3e^0$

$\Rightarrow \quad c_1 + c_2 + c_3 = 0$ ... (4)

(ii) $y'(0) = 0$

$x = 0$ and $y' = 0$

Substituting the corresponding values in equation (2),

$0 = c_1e^0 - 2c_2e^0 + 3c_3e^0$

$\Rightarrow \quad c_1 - 2c_2 + 3c_3 = 0$ ... (5)

(iii) $y''(0) = 1$

$x = 0$ and $y'' = 1$

Substituting the corresponding values in equation (3),

$1 = c_1e^0 + 4c_2e^0 + 9c_3e^0$

$\Rightarrow \quad c_1 + 4c_2 + 9c_3 = 1$ ... (6)

Solving equations (4), (5) and (6),

$$c_1 = \frac{-1}{6}, c_2 = \frac{1}{15}, c_3 = \frac{1}{10}$$

Substituting c_1, c_2 and c_3 values in equation (1),

$$y = \frac{-1}{6}e^x + \frac{1}{15}e^{-2x} + \frac{1}{10}e^{3x}$$

$$\therefore y = -\frac{1}{6}e^x + \frac{1}{5}e^{-2x} + \frac{1}{10}e^{3x}$$

3.2 METHOD OF REDUCTION OF ORDER FOR THE LINEAR HOMOGENEOUS SECOND ORDER DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

Q23. Discuss about the method of reduction of order.

Answer :

A higher order differential equation can be reduced to equation of lower order only when one solution of a differential equation is known. This method is called the reduction of order process. For instance, consider a second-order differential equation of the form.

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y = R(x) \quad \text{... (1)}$$

Let $y_1(x)$ be a solution of equation (1).

Let $y(x) = v(x)\, y_1(x)$ be the general solution where $v(x)$ is to be determined.

$$y(x) = v(x)\, y_1(x) \quad \text{... (2)}$$

Differentiating equation (2) with respect to 'x',

$$\frac{dy}{dx} = v(x)y_1'(x)+v'(x)y_1(x) \quad \text{... (3)}$$

Differentiating equation (3) with respect to 'x',

$$\frac{d^2y}{dx^2} = \left[v(x)y_1''(x)+v'(x)y_1'(x)\right]+\left[v'(x)y_1'(x)+v''(x)y_1(x)\right]$$

$$= v(x)y_1''(x)+2v'(x)y_1'(x)+v''(x)y_1(x) \quad \text{... (4)}$$

Substituting equations (2), (3) and (4) in equation (1)

$$\left[v(x)y_1''(x)+2v'(x)y_1'(x)+v''(x)y_1(x)\right]+P(x)\left[v(x)y_1'(x)+v'(x)y_1(x)\right]+Q(x)v(x)y_1(x)=R(x)$$

$$\Rightarrow \quad v''(x)y_1(x)+v'(x)\left[2y_1'(x)+P(x)y_1(x)\right]+v(x)\left[y_1''(x)+P(x)y_1'(x)+Q(x)y_1(x)\right]=R(x)$$

Since, $y_1(x)$ is a solution of equation (1),

$$\therefore \quad v''(x)y_1(x)+v'(x)\left[2y_1'(x)+P(x)y_1(x)\right]=R(x) \quad \text{... (5)}$$

Substituting $v'(x) = u(x)$ in the equation (5),

$$u'(x)y_1(x)+\left[2y_1'(x)+P(x)y_1(x)\right]u(x)=R(x) \quad \text{... (6)}$$

It can be observed from the equation (6) that, it is a linear differential equation of first order.

It can be solved easily and the general solution can be obtained by integration of $u(x) = v'(x)$. The $v(x)$ value is substituted in $y(x) = v(x)\, y_1(x)$ which is the required solution of differential equation.

Thus, a higher order (i.e, a second order) differential equation can be reduced to lower order (first order) equation.

Q24. If $y_1 = e^{-2x}$ is the one of the solutions of y" – y' – 6y = 0, find other solution by reducing the order of the differential equations.

Answer : June-11, Q13(b)

Given differential equation is,

$$y'' - y' - 6y = 0$$

$$\Rightarrow \quad (D^2 - D - 6) = 0$$

$$\therefore \quad f(D) = D^2 - D - 6$$

The auxiliary equation is $f(m) = 0$

$$\Rightarrow \quad m^2 - m - 6 = 0$$

$$\Rightarrow \quad m^2 - 3m + 2m - 6 = 0$$

$$\Rightarrow \quad (m + 2)(m - 3) = 0$$

$$\Rightarrow \quad m = -2, 3$$

The roots are real and distinct.

$\therefore$ The general solution is $y = c_1 e^{-2x} + c_2 e^{3x}$

Given that, $y_1 = e^{-2x}$ is one solution, the second linearly independent solution is $y_2 = e^{3x}$.

$$\therefore y_2 = e^{3x}$$

Q25. If $y_1 = \frac{1}{x}$ is one of the solutions of the differential equation $x^2y'' + 4xy' + 2y = 0$ then find the general solution of this differential equation by reducing its order.

Dec.-12, Q17(b)

OR

If $\frac{1}{x}$ is a solution of the differential equation $x^2y'' + 4xy' + 2y = 0$. Find the second linearly independent solution.

Answer : Jan.-10, Q12(b))

Given differential equation is,

$$x^2y'' + 4xy' + 2y = 0 \quad \text{... (1)}$$

Equation (1) is of the form $a_0y'' + a_1y' + a_2y = 0$

Since,

$$y_1 = \frac{1}{x}$$

$$P(x) = \frac{a_1(x)}{a_0} = \frac{4x}{x^2} = \frac{4}{x}$$

But, $v(x) = \frac{1}{y_1^2} e^{-\int P(x)dx} = \frac{1}{y_1^2} e^{-\int \frac{4}{x}dx}$

$$v(x) = \frac{1}{\frac{1}{x^2}} e^{-4\log x} = x^2 e^{\log x^{-4}} = x^2 x^{-4}$$

$$\Rightarrow v(x) = \frac{1}{x^2}$$

Let,

$$u(x) = \int v(x)dx$$

$$\Rightarrow u(x) = \int \frac{1}{x^2}dx$$

$$u(x) = \frac{-1}{x}$$

$$y_2(x) = u(x).y_1(x) = \left(\frac{-1}{x}\right)\left(\frac{1}{x}\right)$$

$$y_2(x) = \frac{-1}{x^2}$$

The general solution is $y(x) = Ay_1(x) + By_2(x)$

$$y(x) = \frac{A}{x} - \frac{B}{x^2}$$

As $y_1(x) = 1/x$ is one solution, the second linearly independent solution is $y_2(x) = \frac{-1}{x^2}$

3.3 SOLUTIONS OF NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

Q26. Write the solution of n^{th} order differential equation of the type f(D)y = X.

Answer :

A linear differential equation of the form,

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + ... + a_n y = X \quad \text{... (1)}$$

Where, the right hand side (R.H.S) of the equation is non-zero, is known as n^{th} - order non-homogeneous linear differential equation.

The symbolic form of equation (1) is,

$$(a_0 D^n + a_1 D^{n-1} + \text{- - -} + a_n)y = X \quad \text{... (2)}$$

i.e., $f(D)y = X$

Where,

$f(D)$ – Polynomial of n^{th} order.

X – Function of the form K, e^{ax}, sin ax, cos ax, x^n

The general solution of an n^{th} order non-homogeneous differential equation is,

General solution = Complementary function + Particular integral

i.e., $y = C.F + P.I$

Where,

$$P.I = \frac{X}{f(D)}$$

And,

$$X = K, e^{ax}, \sin ax, \cos ax, x^n$$

Depending on the type of X, the different particular integrals can be obtained as,

S.No.	Function type (X)	Particular Integral (P.I)
1.	Constant = K	$P.I = \frac{1}{f(D)}.K$ Let, D = 0 in f(D) $P.I = \frac{k}{f(0)}$; $f(0) \neq 0$
2.	(a) e^{ax}; $f(a) \neq 0$	$P.I = \frac{1}{f(D)}.e^{ax}$ Let, D = a in f(D) $\Rightarrow P.I = \frac{1}{f(a)}.e^{ax}$

	(b) e^{ax}; f(a) = 0	$P.I = \frac{xe^{ax}}{f'(D)} = \frac{xe^{ax}}{f'(a)}$
	(c) e^{ax}; f'(a) = 0	$P.I = \frac{x^2 e^{ax}}{f''(D)} = \frac{x^2 e^{ax}}{f''(a)}$
3.	(a) sin ax; $f(-a^2) \neq 0$	$P.I = \frac{1}{f(D^2)} \sin ax$ Let, $D^2 = -a^2$ $P.I = \frac{1}{f(-a^2)}.\sin ax$
	(b) sin ax; $f(-a^2) = 0$	$P.I = \frac{-x}{2a}\cos(ax); a \neq 0$
	(c) cos ax; $f(-a^2) \neq 0$	$P.I = \frac{1}{f(D^2)}\cos ax$ Let, $D^2 = -a^2$ $P.I = \frac{1}{f(-a^2)}\cos ax$
	(d) cos ax; $f(-a^2) = 0$	$P.I = \frac{x}{2a}\sin ax; a \neq 0$
4.	x^n	$P.I = \frac{1}{f(D)}x^n = [f(D)]^{-1} x^n$ Where, $[f(D)]^{-1} = [1 \pm g(D)]^{-1}$ is expanded in ascending powers of D, using binomial theorem upto the term D^n. Since $D^{n+1}(x^n) = 0$, $D^{n+2}(x^n) = 0$,..

Q27. Solve $(D^2 - 5D + 6)y = 3e^{5x}$

Answer :

Given differential equation is,

$$(D^2 - 5D + 6)y = 3e^{5x} \quad \text{... (1)}$$

Equation (1) is a non - homogeneous linear differential equation of the form,

$$f(D)y = X \quad \text{... (2)}$$

Comparing equation (2) with equation (1),

$$f(D) = D^2 - 5D + 6 \text{ and } X = 3e^{5x}$$

The general solution of a non-homogeneous linear differential equation is,

$$y = \text{C.F} + \text{P. I} \quad \text{... (3)}$$

The auxiliary equation is, $f(m) = 0$

i.e., $m^2 - 5m + 6 = 0$

$\Rightarrow$ $m^2 - 2m - 3m + 6 = 0$

$\Rightarrow$ $m(m - 2) - 3(m - 2) = 0$

$\Rightarrow$ $(m - 2)(m - 3) = 0$

$\Rightarrow$ $m_1 = 2, m_2 = 3$

$\therefore$ Roots are real and distinct.

Hence the complementary function is,

$$\text{C.F} = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$\Rightarrow \text{C.F} = c_1 e^{2x} + c_2 e^{3x} \quad \text{... (4)}$$

The particular integral is,

$$\text{P.I} = \frac{X}{f(D)}$$

$$= \frac{3e^{5x}}{D^2 - 5D + 6}$$

Here, X is of the form e^{ax}

Let $D = a$ i.e., $D = 5$

$$\Rightarrow \text{P.I} = \frac{3e^{5x}}{5^2 - 5(5) + 6} = \frac{3e^{5x}}{6} = \frac{e^{5x}}{2} \quad \text{... (5)}$$

Substituting equations (4) and (5) in equation (3),

$$\therefore \quad y = c_1 e^{2x} + c_2 e^{3x} + \frac{e^{5x}}{2}.$$

Q28. Solve : $(D^2 + D - 6)y = e^{-2x} + 5$.

Answer :

Given differential equation is,

$$(D^2 + D - 6)y = e^{-2x} + 5 \quad \text{... (1)}$$

Equation (1) is a non-homogeneous linear differential equation of the form,

$$f(D)y = X \quad \text{... (2)}$$

Comparing equation (2) with equation (1),

$$f(D) = D^2 + D - 6 \text{ and } X = e^{-2x} + 5$$

The general solution of a non-homogeneous linear differential equation is,

$$y = \text{C.F} + \text{P.I} \quad \text{... (3)}$$

The auxiliary equation is, $f(m) = 0$

i.e, $m^2 + m - 6 = 0$

$\Rightarrow$ $m^2 + 3m - 2m - 6 = 0$

$\Rightarrow$ $m(m + 3) - 2(m + 3) = 0$

$\Rightarrow$ $(m + 3)(m - 2) = 0$

$\therefore$ $m_1 = 2$ and $m_2 = -3$

The roots are real and distinct

Hence, the complementary function is,

$$\text{C.F} = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$\therefore \quad \text{C.F} = c_1 e^{2x} + c_2 e^{-3x} \quad \text{... (4)}$$

The particular integral is,

$$\text{P.I} = \frac{X}{f(D)} = \frac{e^{-2x}+5}{D^2+D-6}$$

$$= \frac{e^{-2x}}{D^2+D-6} + \frac{5}{D^2+D-6}$$

$\Rightarrow$ $\text{P.I} = \text{P.I}_1 + \text{P.I}_2$ ... (5)

Consider,

$$\text{P.I}_1 = \frac{e^{-2x}}{D^2+D-6}$$

Here, X is of the form e^{ax}

Let, $D = a$ i.e., $D = -2$

$$\Rightarrow \quad \text{P.I}_1 = \frac{e^{-2x}}{(-2)^2+(-2)-6}$$

$$= \frac{e^{-2x}}{4-2-6}$$

$\Rightarrow$ $\text{P.I}_1 = \frac{-e^{-2x}}{4}$ ... (6)

Consider,

$$\Rightarrow \quad \text{P.I}_2 = \frac{5}{D^2+D-6}$$

Here, X is of the form K,

Let, $D = 0$

$$\Rightarrow \quad \text{P.I}_2 = \frac{5}{0+0-6}$$

$\Rightarrow$ $\text{P.I}_2 = -\frac{5}{6}$ ... (7)

Substituting equations (6) and (7) in equation (5),

$\Rightarrow$ $\text{P.I} = \frac{-e^{-2x}}{4} - \frac{5}{6}$ ... (8)

Substituting equations (4) and (8) in equation (3),

$$\therefore \quad y = c_1 e^{2x} + c_2 e^{-3x} - \frac{1}{4}e^{-2x} - \frac{5}{6}$$

Q29. Solve (D^2 – 1)y = cosh 2x.

Answer :

Given differential equation is,

$(D^2 - 1)y = \cosh 2x$ ... (1)

Equation (1) is a non-homogeneous linear differential equation of the form,

$f(D)y = X$ ... (2)

Comparing equations (1) and (2),

$f(D) = D^2 - 1$ and $X = \cosh 2x$

The general solution of a non-homogeneous linear differential equation is,

$y = \text{C.F} + \text{P.I}$ ... (3)

The auxiliary equation is, $f(m) = 0$

i.e., $m^2 - 1 = 0$

$\Rightarrow$ $(m-1)(m+1) = 0$

$\Rightarrow$ $m_1 = 1, m_2 = -1$

The roots are real and distinct.

Hence, the complementary function is,

$$\text{C.F} = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$\Rightarrow$ $\text{C.F} = c_1 e^{x} + c_2 e^{-x}$ ... (4)

The particular integral is,

$$\text{P.I} = \frac{X}{f(D)}$$

$$= \frac{\cosh 2x}{D^2-1}$$

$$= \frac{e^{2x}+e^{-2x}}{2(D^2-1)}$$

$$= \frac{e^{2x}}{2(D^2-1)} + \frac{e^{-2x}}{2(D^2-1)}$$

$\Rightarrow$ $\text{P.I} = \text{P.I}_1 + \text{P.I}_2$ ... (5)

Consider,

$$\text{P.I}_1 = \frac{e^{2x}}{2(D^2-1)}$$

Here, X is of the form e^{ax}

Let, $D = a$ i.e., $D = 2$

$$\Rightarrow \quad \text{P.I}_1 = \frac{e^{2x}}{2((2)^2-1)}$$

$$= \frac{e^{2x}}{2(4-1)}$$

$$\Rightarrow \quad \text{P.I}_1 = \frac{e^{2x}}{6}$$

Consider,

$$\Rightarrow \quad \text{P.I}_2 = \frac{e^{-2x}}{2(D^2-1)}$$

Here, X is of the form e^{ax}

Let, $D = a$ i.e., $D = -2$

$$\Rightarrow \quad \text{P.I}_2 = \frac{e^{-2x}}{2((-2)^2-1)}$$

$$= \frac{e^{-2x}}{2(4-1)}$$

$\Rightarrow$ $\text{P.I}_2 = \frac{e^{-2x}}{6}$ ... (7)

Substituting equations (6) and (7) in equation (5),

$\text{P.I} = \frac{e^{2x}}{6} + \frac{e^{-2x}}{6}$ ... (8)

Substituting equations (4) and (8) in equation (3),

$$y = c_1 e^{x} + c_2 e^{-x} + \frac{e^{2x}}{6} + \frac{e^{-2x}}{6}$$

Q30. Solve (D^2 + 5D + 6)y = sin 5x.

Answer :

Given differential equation is,

$(D^2 + 5D + 6)y = \sin 5x$ ... (1)

Equation (1) is a non-homogeneous linear differential equation of the form,

$$f(D)y = X \quad ...(2)$$

Comparing equation (1) with equation (2),

$$f(D) = D^2 + 5D + 6 \text{ and } X = \sin 5x$$

The general solution of a non-homogeneous linear differential equation is,

$$y = \text{C.F} + \text{P.I} \quad ...(3)$$

The auxiliary equation is $f(m) = 0$

i.e., $m^2 + 5m + 6 = 0$

$\Rightarrow \quad m^2 + 2m + 3m + 6 = 0$

$\Rightarrow \quad m(m + 2) + 3(m + 2) = 0$

$\Rightarrow \quad (m + 2)(m + 3) = 0$

$\Rightarrow \quad m_1 = -2, m_2 = -3$

$\therefore$ Roots are real and distinct

The complementary function is given as,

$$\text{C.F} = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$$\Rightarrow \text{C.F} = c_1 e^{-2x} + c_2 e^{-3x} \quad ...(4)$$

The particular integral is given as,

$$\text{P.I} = \frac{X}{f(D)} = \frac{\sin 5x}{D^2 + 5D + 6}$$

Here, X is of the form, $\sin ax$

Let $D^2 = -a^2$ i.e., $D^2 = -5^2 = -25$

$$\text{P.I} = \frac{\sin 5x}{-25 + 5D + 6}$$

$$= \frac{\sin 5x}{-19 + 5D}$$

$$= \frac{\sin 5x}{5D - 19} \times \frac{5D + 19}{5D + 19}$$

$$= \frac{5.D(\sin 5x) + 19 \sin 5x}{(5D)^2 - 19^2}$$

$$= \frac{5 \cos 5x(5) + 19 \sin 5x}{25D^2 - 361}$$

$$= \frac{25 \cos 5x + 19 \sin 5x}{25(-25) - 361}$$

$$= \frac{25 \cos 5x + 19 \sin 5x}{-986}$$

$$\Rightarrow \text{P.I} = \frac{-1}{986}[25 \cos 5x + 19 \sin 5x] \quad ...(5)$$

Substituting equations (4) and (5) in equation (3),

$$\therefore y = c_1 e^{-2x} + c_2 e^{-3x} - \frac{1}{986}[25 \cos 5x + 19 \sin 5x]$$

Q31. Solve : $(D^2 + 8)y = \sin^2 x$.

Answer :

Given differential equation is,

$$(D^2 + 8)y = \sin^2 x \quad ...(1)$$

Equation (1) is a non-homogeneous linear differential equation of the form,

$$f(D)y = X \quad ...(2)$$

Comparing equation (2) with equation (1),

$$f(D) = D^2 + 8 \text{ and } X = \sin^2 x$$

The general solution of a non-homogeneous linear differential equation is,

$$y = \text{C.F} + \text{P.I} \quad ...(3)$$

The auxiliary equation is, $f(m) = 0$

i.e, $\quad m^2 + 8 = 0$

$\Rightarrow \quad m^2 = -8$

$\therefore \quad m = \pm 2\sqrt{2}\,i$

The roots are complex conjugate of the form $\alpha \pm i\beta$

Here $\alpha = 0, \beta = 2\sqrt{2}$

Hence, the complementary function is,

$$\text{C.F} = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin \beta x)$$

$$= e^{0x}(c_1 \cos 2\sqrt{2}x + c_2 \sin 2\sqrt{2}x)$$

$$\therefore \text{C.F} = c_1 \cos 2\sqrt{2}x + c_2 \sin 2\sqrt{2}x \quad ...(4)$$

The particular integral is,

$$\text{P.I} = \frac{X}{f(D)} = \frac{\sin^2 x}{D^2 + 8}$$

$$= \frac{1 - \cos 2x}{2(D^2 + 8)} \qquad [\because \cos x = 2\sin^2 x - 1]$$

$$= \frac{1}{2(D^2 + 8)} - \frac{\cos 2x}{2(D^2 + 8)}$$

$$\Rightarrow \text{P.I} = \text{P.I}_1 + \text{P.I}_2 \quad ...(5)$$

Consider,

$$\text{P.I}_1 = \frac{1}{2(D^2 + 8)}$$

Here, X is of the form K,

Let $D = 0$

$$\text{P.I}_1 = \frac{1}{2(0^2 + 8)} = \frac{1}{16} \quad ...(6)$$

Consider,

$$\text{P.I}_2 = \frac{-\cos 2x}{2(D^2 + 8)}$$

Here, X is of the form $\cos ax$,

Let, $D^2 = -a^2$ i.e., $D^2 = -2^2 = -4$

$$\Rightarrow \text{P.I}_2 = \frac{\cos 2x}{2(-4 + 8)} = \frac{\cos 2x}{8} \quad ...(7)$$

Substituting equations (6) and (7) in equation (5),

$$\text{P.I} = \frac{1}{16} - \frac{\cos 2x}{8} \quad ...(8)$$

Substituting equations (4) and (8) in equation (3),

$\therefore \quad y = c_1\cos 2\sqrt{2}\,x + c_2\sin 2\sqrt{2}\,x + \frac{1}{16} - \frac{\cos 2x}{8}$

Q32. Solve $(D^2 + 5D + 6)y = e^{2x} + \sin x$.

Answer : Model Paper-3, Q13(a)

Given differential equation is,

$(D^2 + 5D + 6)y = e^{2x} + \sin x$ ... (1)

Equation (1) is the non-homogeneous linear differential equation of the form,

$f(D)y = X$ ... (2)

Comparing equations (1) and (2),

$f(D) = D^2 + 5D + 6$ and $X = e^{2x} + \sin x$

The general solution of a non-homogeneous linear differential equation is,

$y = \text{C.F} + \text{P.I}$ ... (3)

The auxiliary equation is, $f(m) = 0$,

i.e., $m^2 + 5m + 6 = 0$

$\Rightarrow \quad m^2 + 3m + 2m + 6 = 0$

$\Rightarrow \quad m(m + 3) + 2(m + 3) = 0$

$\Rightarrow \quad (m + 2)(m + 3) = 0$

$\Rightarrow \quad m_1 = -2, m_2 = -3$

$\therefore$ The roots are real and distinct

The complementary function is,

$\text{C.F} = c_1 e^{-2x} + c_2 e^{-3x}$ ... (4)

The particular integral is,

$$\text{P.I} = \frac{X}{f(D)}$$

$$= \frac{e^{2x} + \sin x}{D^2 + 5D + 6}$$

$$= \frac{e^{2x}}{D^2 + 5D + 6} + \frac{\sin x}{D^2 + 5D + 6}$$

$\Rightarrow \quad \text{P.I} = \text{P.I}_1 + \text{P.I}_2$ ... (5)

Consider,

$$\text{P.I}_1 = \frac{e^{2x}}{D^2 + 5D + 6}$$

Here, X is of the form e^{ax}

Let, $D = a$ i.e., $D = 2$

$$\text{P.I}_1 = \frac{e^{2x}}{(2)^2 + 5(2) + 6}$$

$$= \frac{e^{2x}}{4 + 10 + 6}$$

$\Rightarrow \quad \text{P.I}_1 = \frac{e^{2x}}{20}$ ... (6)

Consider,

$$\text{P.I}_2 = \frac{\sin x}{D^2 + 5D + 6}$$

Here, X is of the form $\sin ax$,

Let Let, $D^2 = -a^2$ i.e., $D^2 = -1^2 = -1$

$$\text{P.I}_2 = \frac{\sin x}{-1 + 5D + 6}$$

$$= \frac{\sin x}{5D + 5} \times \frac{5D - 5}{5D - 5}$$

$$= \frac{5D(\sin x) - 5\sin x}{(5D)^2 - (5)^2}$$

$$= \frac{5.\cos x - 5\sin x}{25D^2 - 25}$$

$$= \frac{5[\cos x - \sin x]}{25(-1) - 25}$$

$$= \frac{5(\cos x - \sin x)}{-25 - 25}$$

$$= \frac{5(\cos x - \sin x)}{-50}$$

$\Rightarrow \quad \text{P.I} = -\frac{1}{10}(\cos x - \sin x)$ ... (7)

Substituting equations (6) and (7) in equation (5),

$\text{P.I} = \frac{e^{2x}}{20} - \frac{1}{10}(\cos x - \sin x)$ ... (8)

Substituting equations (4) and (8) in equation (3),

$\therefore \quad y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{e^{2x}}{20} - \frac{1}{10}(\cos x - \sin x)$

Q33. Solve : $(D^2 + 3D + 2)y = 5 + 2x$.

Answer :

Given differential equation is,

$(D^2 + 3D + 2)y = 5 + 2x$ ... (1)

Equation (1) is a non-homogeneous linear differential equation of the form,

$f(D)y = X$ ... (2)

Comparing equation (2) with equation (1),

$f(D) = D^2 + 3D + 2, \; X = 5 + 2x$

The general solution of a non-homogeneous linear differential equation is,

$y = \text{C.F} + \text{P.I}$ ... (3)

The auxiliary equation is, $f(m) = 0$

i.e, $m^2 + 3m + 2 = 0$

$\Rightarrow m^2 + 2m + m + 2 = 0$

$\Rightarrow (m + 2)(m + 1) = 0$

$\Rightarrow m_1 = -1, m_2 = -2$

The roots are real and distinct

Hence, the complementary function is,

$\text{C.F} = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\Rightarrow \quad \text{C.F} = c_1 e^{-x} + c_2 e^{-2x}$ ... (4)

The particular integral is,

$$\text{P.I} = \frac{X}{f(D)} = \frac{5+2x}{D^2+3D+2}$$

$$= \frac{5}{D^2+3D+2} + \frac{2x}{D^2+3D+2}$$

$\Rightarrow$ $\text{P.I} = \text{P.I}_1 + \text{P.I}_2$ ... (5)

Consider,

$$\text{P.I}_1 = \frac{5}{D^2+3D+2}$$

Here, X is of the form K

Let, $D = 0$

$$\text{P.I}_1 = \frac{5}{0^2+3(0)+2}$$

$\Rightarrow$ $\text{P.I}_1 = \frac{5}{2}$ ... (6)

Consider,

$$\text{P.I}_2 = \frac{2x}{D^2+3D+2}$$

$$= \frac{2x}{2\left[1+\frac{D^2+3D}{2}\right]}$$

$$= x\left[1+\frac{D^2+3D}{2}\right]^{-1} \qquad [\because (1+D)^{-1} = 1 - D + D^2 - D^3 + ...]$$

$$= x\left[1-\left(\frac{D^2+3D}{2}\right)+\left(\frac{D^2+3D}{2}\right)^2 - ......\right]$$

$$= x\left[1-\frac{D^2}{2}-\frac{3D}{2}+\frac{D^4}{4}+\frac{9D^2}{4}+\frac{6D^3}{4} - .....\right]$$

$$= \left[x-\frac{D^2}{2}(x)-\frac{3D}{2}(x)+\frac{D^4}{4}(x)+\frac{9D^2}{4}(x)+\frac{6D^3}{4}(x) -....\right]$$

$$= x - \frac{3}{2}D(x)$$

$$= x - \frac{3}{2}\ (1)$$

$\Rightarrow$ $\text{P.I}_2 = x - \frac{3}{2}$ ... (7)

Substituting equations (6) and (7) in equation (5),

$$\text{P.I} = \frac{5}{2} + x - \frac{3}{2}$$

$\Rightarrow$ $\text{P.I} = x + 1$ ... (8)

Substituting equations (4) and (8) in equation (3),

$\therefore$ $y = c_1 e^{-x} + c_2 e^{-2x} + x + 1$

Q34. Solve : $(D^2 + 1)y = x^2 + 3x$.

Answer :

Given differential equation is,

$(D^2 + 1)y = x^2 + 3x$ ... (1)

Equation (1) is a non-homogeneous linear differential equation of the form,

$f(D)y = X$ ... (2)

Comparing equation (2) with equation (1),

$$f(D) = D^2 + 1 \text{ and } X = x^2 + 3x$$

The general solution of a non-homogeneous linear differential equation is given as,

$$y = \text{C.F} + \text{P.I} \qquad \text{... (3)}$$

The auxiliary equation is, $f(m) = 0$,

i.e, $m^2 + 1 = 0$

$\Rightarrow$ $m^2 = -1$

$\Rightarrow$ $m = \pm i$

The roots are complex conjugate of the form $\alpha \pm i\beta$;

Here $\alpha = 0, \beta = 1$

Hence, the complementary function is,

$$\text{C.F} = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

$$= e^{0x}(c_1 \cos x + c_2 \sin x)$$

$\therefore$ $\text{C.F} = c_1\cos x + c_2\sin x$ ... (4)

The particular integral is,

$$\text{P.I} = \frac{X}{f(D)}$$

$$= \frac{x^2 + 3x}{D^2 + 1}$$

$$= \frac{x^2}{D^2+1} + \frac{3x}{D^2+1}$$

$\Rightarrow$ $\text{P.I} = \text{P.I}_1 + \text{P.I}_2$ ... (5)

Consider,

$$\text{P.I}_1 = \frac{x^2}{D^2+1}$$

$$= x^2[1 + D^2]^{-1}$$

$$= x^2[1 - D^2 + D^4 - D^6 + ...]$$

$$= x^2 - D^2(x^2) + D^4(x^2) - D^6(x^2) + ...$$

$\Rightarrow$ $\text{P.I}_1 = x^2 - D^2(x^2)$

$\Rightarrow$ $\text{P.I}_1 = x^2 - 2$ ... (6)

Consider,

$$\text{P.I}_2 = \frac{3x}{D^2+1}$$

$$= 3x[1 + D^2]^{-1}$$

$$= 3x[1 - D^2 + D^4 - D^6 + ...]$$

$$= 3[x - D^2(x) + D^4(x) - D^6(x) + ...]$$

$\Rightarrow$ $\text{P.I}_2 = 3x$ ... (7)

Substituting equations (6) and (7) in equation (5),

$\Rightarrow$ $\text{P.I} = x^2 - 2 + 3x$ ... (8)

Substituting equations (4) and (8) in equation (3),

$\therefore$ $y = c_1 \cos x + c_2 \sin x + x^2 - 2 + 3x.$

Q35. Solve $(D^2 + D - 2)y = x + \sin x$.

Answer :

Given differential equation is,

$$(D^2 + D - 2)y = x + \sin x \qquad \text{... (1)}$$

Equation (1) is a non-homogeneous linear differential equation of the form

$$f(D)y = X \qquad \text{... (2)}$$

Comparing equation (2) with equation (1),

$$f(D) = (D^2 + D - 2) \text{ and } X = x + \sin x$$

The general solution of a non-homogeneous linear differential equation is,

$$y = \text{C.F} + \text{P.I} \qquad \text{... (3)}$$

The auxiliary equation is, $f(m) = 0$

i.e., $m^2 + m - 2 = 0$

$\Rightarrow$ $m^2 + 2m - m - 2 = 0$

$\Rightarrow$ $m(m + 2) - 1(m + 2) = 0$

$\Rightarrow$ $(m + 2)(m - 1) = 0$

$\Rightarrow$ $m_1 = -2;\ m_2 = 1$

$\therefore$ The roots are real and distinct.

Hence the complementary function is,

$$\text{C.F} = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

$\therefore$ $\text{C.F} = c_1 e^{-2x} + c_2 e^{x}$ ... (4)

The particular integral is,

$$\text{P.I} = \frac{X}{f(D)} = \frac{x + \sin x}{D^2 + D - 2}$$

$$= \frac{x}{D^2 + D - 2} + \frac{\sin x}{D^2 + D - 2}$$

$\Rightarrow$ $\text{P.I} = \text{P.I}_1 + \text{P.I}_2$ ... (5)

Consider,

$$\text{P.I}_1 = \frac{x}{D^2 + D - 2}$$

$$= \frac{x}{-2\left[1 - \frac{(D^2 + D)}{2}\right]}$$

$$= \frac{-x}{2}\left[1 - \left(\frac{D^2 + D}{2}\right)\right]^{-1}$$

$$\left[\because (1 - D)^{-1} = 1 + D + D^2 + D^3 + ...\right]$$

$$= \frac{-x}{2}\left[1 + \left(\frac{D^2 + D}{2}\right) + \left(\frac{D^2 + D}{2}\right)^2 + ...\right]$$

$$= \frac{-x}{2}\left[1+\frac{D^2}{2}+\frac{D}{2}+\frac{D^4}{4}+\frac{2D^3}{4}+\frac{D^4}{4}+.....\right]$$

$$= \frac{-1}{2}\left[x+\frac{D^2}{2}(x)+\frac{D}{2}(x)+\frac{D^4}{2}(x)+\frac{2D^3}{4}(x)+\frac{D^2}{4}(x)+.....\right]$$

$$= \frac{-1}{2}\left[x+\frac{D}{2}(x)\right]$$

$$= \frac{-1}{2}\left[x+\frac{1}{2}(1)\right]$$

$\Rightarrow \quad P.I = \frac{-x}{2}-\frac{1}{4}$ ... (6)

Consider

$$P.I_2 = \frac{\sin x}{D^2+D-2}$$

Here, X is of the form sin ax

Let $D^2 = -a^2$ i.e., $D^2 = -(1)^2 = -1$

$$P.I_2 = \frac{\sin x}{-1+D-2}$$

$$= \frac{\sin x}{D-3}\times\frac{D+3}{D+3}$$

$$= \frac{\sin x\,(D+3)}{D^2-9}$$

$$= \frac{D(\sin x)+3\sin x}{-1-9}$$

$$= \frac{\cos x+3\sin x}{-10}$$

$\Rightarrow \quad P.I_2 = \frac{-1}{10}\left[3\sin x+\cos x\right]$ ... (7)

Substituting equations (6) and (7) in equation (5),

$P.I = \frac{-x}{2}-\frac{1}{4}-\frac{1}{10}\left[3\sin x+\cos x\right]$ ... (8)

Substituting equations (4) and (8) in equation (3),

$$\therefore \quad y = c_1e^{-2x}+c_2e^{x}-\frac{x}{2}-\frac{1}{4}-\frac{1}{10}\left[3\sin x+\cos x\right]$$

Q36. Solve $(D^2 – 4D + 4)y = 8(x^2 + e^{2x} + \sin 2x)$.

Answer :

Given differential equation is,

$(D^2 - 4D + 4)y = 8(x^2 + e^{2x} + \sin 2x)$ ... (1)

Equation (1) is a non-homogeneous linear differential equation of the form,

$f(D)y = X$ ... (2)

Comparing equations (1) and (2),

$f(D) = D^2 - 4D + 4$

$X = 8(x^2 + e^{2x} + \sin 2x)$

The general solution is,

$y = y_c + y_p$ ... (3)

The auxiliary equation is, $f(m) = 0$

$\Rightarrow \quad m^2 - 4m + 4 = 0$

$\Rightarrow \quad (m-2)^2 = 0$

$\Rightarrow \quad m = 2, 2$

The roots are real and equal

Complementary function is, $y_c = [c_1 + c_2x]e^{mx}$

$\Rightarrow \quad y_c = [c_1 + c_2x]e^{2x}$ ... (4)

Particular integral is,

$$y_p = \frac{X}{f(D)} = \frac{1}{(D^2-4D+4)} \, 8(x^2 + e^{2x} + \sin 2x)$$

$$\Rightarrow \quad y_p = 8\left[\frac{x^2}{D^2-4D+4} + \frac{e^{2x}}{D^2-4D+4} + \frac{\sin 2x}{D^2-4D+4}\right] \quad \text{... (5)}$$

Consider,

$$\frac{x^2}{D^2-4D+4} = \frac{x^2}{4\left[1+\frac{D^2-4D}{4}\right]}$$

$$= \frac{1}{4}\left[1+\frac{D^2-4D}{4}\right]^{-1} x^2$$

$$= \frac{1}{4}\left[1-\left(\frac{D^2-4D}{4}\right)+\left(\frac{D^2-4D}{4}\right)^2 \ldots\ldots\right] x^2$$

$$= \frac{1}{4}\left[1-\frac{D^2}{4}+\frac{4D}{4}+\frac{16D^2}{16}-\frac{8D^3}{16}\right]x^2$$

$$= \frac{1}{4}\left[1(x^2)-\frac{D^2}{4}(x^2)+\frac{4D}{4}(x^2)+\frac{16}{16}D^2(x^2)-\frac{8D^3}{16}(x^2)\right]$$

$$= \frac{1}{4}\left[x^2-\frac{1}{4}(2)+2x+2-0\right]$$

$$= \frac{1}{4}\left[x^2-\frac{1}{2}+2x+2\right]$$

$$= \frac{1}{4}\left[x^2+2x+\frac{3}{2}\right]$$

$$\Rightarrow \quad \frac{x^2}{D^2-4D+4} = \frac{1}{4}\left[x^2+2x+\frac{3}{2}\right]$$

Consider,

$$\frac{e^{2x}}{D^2-4D+4} = \frac{e^{2x}}{(D-2)^2}$$

$$= \frac{e^{2x}.x^2}{2!} \qquad \left[\because \frac{e^{ax}}{(D-a)^n} = \frac{x^n}{n!}e^{ax}\right]$$

$$= \frac{x^2e^{2x}}{2}$$

$$\Rightarrow \quad \frac{e^{2x}}{D^2-4D+4} = \frac{x^2e^{2x}}{2}$$

Consider,

$$\frac{\sin 2x}{D^2-4D+4}$$

Here, X is of the form $\sin ax$

Let, $D^2 = -a^2$ i.e., $D^2 = -2^2 = -4$

$$\Rightarrow \quad \frac{\sin 2x}{D^2 - 4D + 4} = \frac{\sin 2x}{-4 - 4D + 4}$$

$$= \frac{\sin 2x}{-4D}$$

$$= \frac{-1}{4} \cdot \frac{1}{D}(\sin 2x)$$

$$= \frac{-1}{4}\int \sin 2x dx$$

$$= \frac{-1}{4}\left(\frac{-\cos 2x}{2}\right)$$

$$= \frac{\cos 2x}{8}$$

$$\therefore \quad \frac{\sin 2x}{D^2 - 4D + 4} = \frac{\cos 2x}{8}$$

Substituting the corresponding values in equation (5)

$$y_p = 8\left[\frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right) + \frac{x^2 e^{2x}}{2} + \frac{\cos 2x}{8}\right]$$

$$= \frac{8}{4}\left(x^2 + 2x + \frac{3}{2}\right) + \frac{x^2 e^{2x}.8}{2} + \frac{8\cos 2x}{8}$$

$$y_p = 2\left(x^2 + 2x + \frac{3}{2}\right) + 4x^2 e^{2x} + \cos 2x \quad \text{... (6)}$$

Substituting equations (4) and (6) in equation (3),

$$\therefore \quad y = (c_1 + c_2 x)e^{2x} + 2\left(x^2 + 2x + \frac{3}{2}\right) + 4x^2 e^{2x} + \cos 2x$$

Q37. Solve $(D^3 + 1)\,y = \cos(2x - 1) + x^2 e^{-x}$.

Answer : **Model Paper-3, Q13(b)**

Given differential equation is,

$$(D^3 + 1)y = \cos(2x - 1) + x^2 e^{-x} \quad \text{... (1)}$$

Equation (1) is of the form

$$f(D)y = X \quad \text{... (2)}$$

Comparing equations (1) and (2),

$$f(D) = D^3 + 1 \text{ and } X = \cos(2x - 1) + x^2 e^{-x}$$

Auxiliary equation is, $f(m) = 0$

$$\Rightarrow \quad m^3 + 1 = 0$$

$$\Rightarrow \quad (m + 1)(m^2 - m + 1) = 0$$

$$\Rightarrow \quad m = -1, \ \frac{1 \pm i\sqrt{3}}{2}$$

The complementary function is given as,

$$\text{C.F} = c_1 e^{-x} + e^{\frac{x}{2}}\left[c_2 \cos\sqrt{\frac{3}{2}}x + c_3 \sin\sqrt{\frac{3}{2}}x\right]$$

The particular integral is given as,

$$\text{P.I} = \frac{1}{f(D)} X$$

$$= \frac{1}{D^3 + 1}\left(\cos(2x - 1) + x^2 e^{-x}\right)$$

$$\Rightarrow \quad \text{P.I} = \frac{\cos(2x - 1)}{D^3 + 1} + \frac{x^2 e^{-x}}{D^3 + 1} \quad \text{... (3)}$$

Consider,

$$\text{P.I}_1 = \frac{1}{D^3 + 1}\cos(2x - 1)$$

Substituting $D^2 = -4$ in above equation,

$$\text{P.I}_1 = \frac{1}{1 - 4D}\cos(2x - 1)$$

$$\Rightarrow \quad \text{P.I}_1 = \frac{1 + 4D}{(1 + 4D)(1 - 4D)}\cos(2x - 1)$$

$$\Rightarrow \quad \text{P.I}_1 = \frac{1 + 4D}{1 - 16D^2}\cos(2x - 1)$$

Substituting $D^2 = -4$ in above equation,

$$\text{P.I}_1 = \frac{1 + 4D}{1 - 16(-4)}(\cos(2x - 1))$$

$$\Rightarrow \quad \text{P.I}_1 = \frac{1}{65}[\cos(2x - 1) + 4D(\cos(2x - 1))]$$

$$\Rightarrow \quad \text{P.I}_1 = \frac{1}{65}[\cos(2x - 1) + 4 \times 2(-\sin(2x - 1))]$$

$$\therefore \quad \text{P.I}_1 = \frac{1}{65}[\cos(2x - 1) - 8\sin(2x - 1)] \quad \text{... (4)}$$

Consider,

$$\text{P.I}_2 = \frac{x^2 e^{-x}}{(D^3 + 1)}$$

$$= \frac{e^{-x}x^2}{(D - 1)^3 + 1}$$

$$= \frac{e^{-x}x^2}{D^3 - 3D^2 + 3D - 1 + 1}$$

$$= \frac{e^{-x}x^2}{D^3 - 3D^2 + 3D}$$

$$= \frac{e^{-x}x^2}{D(D^2 - 3D + 3)}$$

$$= \frac{e^{-x}x^2}{3D\left[1 + \frac{D^2 - 3D}{3}\right]}$$

$$= \frac{e^{-x}x^2}{3D}\left[1 + \frac{D^2 - 3D}{3}\right]^{-1}$$

$$= \frac{e^{-x}}{3D}\left[1 - \frac{D^2 - 3D}{3} + \left(\frac{D^2 - 3D}{3}\right)^2 - \left(\frac{D^2 - 3D}{3}\right)^3 + ...\right]x^2$$

$$= \frac{e^{-x}}{3D}\left[1 - \frac{(D^2 - 3D)}{3} + \frac{D^4 - 6D^3 + 9D^2}{9}\right]x^2$$

$$= \frac{e^{-x}}{3D}\left[x^2 - \frac{D^2x^2}{3} + \frac{3Dx^2}{3} + \frac{D^4x^2 - 6D^3x^2 + 9D^2x^2}{9}\right]$$

$$= \frac{e^{-x}}{3D}\left[x^2 - \frac{2Dx}{3} + 2x + 0 - 0 + 2Dx\right]$$

$$= \frac{e^{-x}}{3D}\left[x^2 - \frac{2}{3} + 2x + 2\right]$$

$$= \frac{e^{-x}}{3D}\left[x^2 + 2x + \frac{4}{3}\right]$$

$$= \frac{e^{-x}}{3}\int\left[x^2 + 2x + \frac{4}{3}\right]dx$$

$$= \frac{e^{-x}}{3}\left[\frac{x^3}{3} + \frac{2x^2}{2} + \frac{4x}{3}\right]$$

$$= \frac{e^{-x}}{3}\left[\frac{x^3}{3} + x^2 + \frac{4x}{3}\right]$$

$$= \frac{e^{-x}}{9}[x^3 + 3x^2 + 4x]$$

$\therefore \quad P.I_2 = \frac{e^{-x}}{9}[x^3 + 3x^2 + 4x]$ ... (5)

Substituting equations (4) and (5) in equation (3),

$P.I = \frac{1}{65}[\cos(2x - 1) - 8\sin(2x - 1)] + \frac{e^{-x}}{9}[x^3 + 3x^2 + 4x]$ ... (6)

The general solution is given as,

$y = C.F + P.I$ ... (7)

Substituting the corresponding values in equation (7)

$$\therefore \quad y = c_1 e^{-x} + e^{\frac{x}{2}}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right] + \frac{1}{65}[\cos(2x - 1) - 8\sin(2x - 1)] + \frac{e^{-x}}{9}[x^3 + 3x^2 + 4x]$$

Q38. Solve (D^2 + 2D + 1) y = xe^x cos x.

Answer : Model Paper-2, Q16(b)

Given differential equation is,

$(D^2 + 2D + 1)\, y = x\, e^x \cos x$

Let, $f(D) = D^2 + 2D + 1$ and $X = xe^x \cos x$

The auxiliary equation is,

$f(m) = 0$

$\Rightarrow \quad m^2 + 2m + 1 = 0$

$\Rightarrow \quad (m + 1)^2 = 0$

$\Rightarrow \quad m = -1, -1$

The roots are real and equal.

The complementary function (C.F) is,

$y_c = (c_1 + c_2 x)\, e^{-x}$

The particular integral (P.I) is given by,

$$y_p = \frac{1}{f(D)}.X$$

$$= \frac{1}{D^2+2D+1}.xe^x \cos x$$

$$= \frac{1}{(D+1)^2}.xe^x \cos x$$

$$= e^x.\frac{x\cos x}{(D+1+1)^2} = e^x.\frac{x\cos x}{(D+2)^2}$$

$$= e^x.\left[x - \frac{2(D+2)}{(D+2)^2}\right]\frac{\cos x}{(D+2)^2}$$

$$= e^x.\left[x - \frac{2}{D+2}\right]\frac{\cos x}{D^2+4+4D}$$

$$= e^x\left[x - \frac{2}{D+2}\right]\frac{\cos x}{-1+4+4D} = e^x\left[x - \frac{2}{D+2}\right]\frac{\cos x}{4D+3}$$

$$= e^x\left[x - \frac{2}{D+2}\right]\frac{(4D-3)}{16D^2-9}\cos x = e^x\left[x - \frac{2}{D+2}\right]\frac{(4D-3)\cos x}{-25}$$

$$= \frac{e^x}{-25}\left[x - \frac{2}{D+2}\right](4(-\sin x) - 3\cos x)$$

$$= \frac{e^x}{-25}\left[x - \frac{2}{D+2}\right](-4\sin x - 3\cos x)$$

$$= \frac{e^x}{-25}x(-4\sin x - 3\cos x) + \frac{2e^x}{25}\left(\frac{-4\sin x - 3\cos x}{D+2}\right)$$

$$= \frac{e^x.x}{25}(4\sin x + 3\cos x) - \frac{2e^x}{25}\frac{(4\sin x + 3\cos x)}{D+2}$$

$$= \frac{e^x x}{25}(4\sin x + 3\cos x) - \frac{2e^x}{25}\frac{2e^x(D-2)(4\sin x + 3\cos x)}{D^2-4}$$

$$= \frac{e^x.x}{25}(4\sin x + 3\cos x) - \frac{2e^x}{25}\frac{(D-2)(4\sin x + 3\cos x)}{(-1-4)}$$

$$= \frac{e^x.x}{25}(4\sin x + 3\cos x) + \frac{2e^x}{125}(D-2)(4\sin x + 3\cos x)$$

$$= \frac{e^x.x}{25}(4\sin x + 3\cos x) + \frac{2e^x}{125}(4\cos x - 3\sin x - 8\sin x - 6\cos x)$$

$$= \frac{e^x.x}{25}(4\sin x + 3\cos x) + \frac{2e^x}{125}(-2\cos x - 11\sin x)$$

$$\therefore \quad y_p = \frac{e^x.x}{25}(4\sin x + 3\cos x) - \frac{2e^x}{125}(2\cos x + 11\sin x)$$

The general solution is given by,

$$y = y_c + y_p$$

$$\therefore \quad y = (c_1 + c_2 x)\, e^{-x} + \frac{e^x . x}{25}(4\sin x + 3\cos x) - \frac{2e^x}{125}(2\cos x + 11\sin x)$$

3.4 METHOD OF VARIATION OF PARAMETERS, SOLUTION OF EULER-CAUCHY EQUATION

Q39. Determine the solution of second order differential equation using method of variation of parameters.

Answer :

A second order linear differential equation with constant coefficients is expressed as,

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = Q(x)$$

Where,

a_1, a_2 are constant coefficients

$Q(x)$ is a continuous function of x

$Q(x) \neq 0$ and $a_2 \neq 0$

The complementary function is given as,

$$\text{C.F} = c_1 f_1 + c_2 f_2$$

Where,

c_1, c_2 are constant and f_1, f_2 are functions of x

The particular integral is given as,

$$\text{P.I} = Pf_1 + Rf_2$$

Where,

$$P = -\int \frac{f_2 Q(x)}{f_1 f_2' - f_1' f_2} dx$$

$$R = \int \frac{f_1 Q(x)}{f_1 f_2' - f_1' f_2} dx$$

$\therefore$ The general solution is,

$$y = \text{C.F} + \text{P.I}$$

$$\Rightarrow \quad y = c_1 f_1 + c_2 f_2 + Pf_1 + Rf_2.$$

Q40. Solve (D^2 + 1) y = sec^2 x by the method of variation parameters.

Answer :

Given differential equation is,

$$(D^2 + 1)y = \sec^2 x \quad \text{... (1)}$$

Equation (1) is a non homogeneous linear differential equation of the form,

$$f(D)y = Q(x) \quad \text{... (2)}$$

Comparing equations (1) and (2),

$$f(D) = D^2 + 1,\ Q(x) = \sec^2 x$$

General solution is given as,

$$y = y_c + y_p \quad \text{... (3)}$$

Auxiliary equation is,

$f(m) = 0$

i.e., $m^2 + 1 = 0$

$\Rightarrow \quad m = \pm i$

Roots are complex conjugate of the form $\alpha + i\beta$

The complementary function (C.F) is given as,

$$y_c = (c_1 \cos x + c_2 \sin x) \quad \text{... (4)}$$

Particular integral (P.I) is given as,

$$y_p = Pf_1 + Rf_2 \quad \text{... (5)}$$

Where,

$$f_1 = \cos x, f_2 = \sin x$$

$$f_1' = -\sin x,\ f_2' = \cos x$$

$$f_1 f_1' - f_1'\ f_2 = \cos^2 x + \sin^2 x = 1$$

$$P = -\int \frac{f_2 Q(x)}{f_1 f_2' - f_1' f_2}\, dx$$

$$= -\int \frac{\sin x \sec^2 x}{1} dx$$

$$= -\int \frac{\sin x}{\cos x} . \sec x\, dx$$

$$= -\int \tan x \sec x\, dx$$

$$= -\sec x$$

$$R = \int \frac{f_1 Q(x)}{f_1 f_2' - f_1' f_2}\, dx$$

$$= \int \frac{\cos x \sec^2 x}{1} dx$$

$$= \int \sec x\, dx$$

$$= \log(\sec x + \tan x)$$

Substituting the corresponding values in equation (5),

$$y_p = (-\sec x)\cos x + [\log(\sec x + \tan x)] \sin x$$

$$\therefore \quad y_p = -1 + \sin x \log(\sec x + \tan x) \quad \text{... (6)}$$

Substituting equations (4) and (6) in equation (3),

$$y = c_1 \cos x + c_2 \sin x - 1 + \sin x [\log(\sec x + \tan x)] + c$$

$$\therefore \quad y = c_1 \cos x + c_2 \sin x - 1 + \sin x [\log(\sec x + \tan x)] + c.$$

Q41. Solve $\frac{d^2y}{dx^2} + y = \text{cosec}x$ by using method of variation of parameters.

Answer :

Given differential equation is,

$$\frac{d^2y}{dx^2} + y = \text{cosec } x$$

$$\Rightarrow \quad (D^2 + 1)\, y = \text{cosec } x \quad \text{... (1)}$$

Equation (1) is a non-homogeneous linear differential equation of the form,

$$f(D)y = Q(x) \quad \text{... (2)}$$

Comparing equations (1) and (2)

$$f(D) = D^2 + 1$$

$$Q(x) = \text{cosec } x$$

The general solution is given as,

$$y = y_c + y_p \quad \text{... (3)}$$

The auxiliary equation is,

$f(m) = 0$

$\Rightarrow \quad m^2 + 1 = 0$

$\Rightarrow \quad m^2 = -1$

$\Rightarrow \quad m = \pm i$

The roots are complex conjugate of the form $\alpha \pm i\beta$.

The complementary function is given as

$y_c = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

Here $\alpha = 0$; $\beta = 1$

$y_c = e^0 (c_1 \cos x + c_2 \sin x)$

$y_c = c_1 \cos x + c_2 \sin x$ ... (4)

The particular integral is given by

$y_p = Pf_1 + Rf_2$ ... (5)

Where, $f_1 = \cos x$; $f_2 = \sin x$

$f_1' = -\sin x$; $f_2' = \cos x$

$f_1 f_2' - f_1' f_2 = \cos x(\cos x) - (-\sin x)(\sin x)$

$= \cos^2 x + \sin^2 x$

$= 1$

$$P = -\int \frac{f_2 Q(x)}{f_1 f_2' - f_1' f_2} dx$$

$$= -\int \frac{\sin x(\operatorname{cosec} x)}{1} dx$$

$$= -\int \sin x . \operatorname{cosec} x \, dx$$

$$= -\int \sin x . \frac{1}{\sin x} dx$$

$$= -\int dx$$

$= -x$

$\Rightarrow \quad P = -x$ ... (6)

$$R = \int \frac{f_1 Q(x)}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\cos x . (\operatorname{cosec} x)}{1} dx$$

$$= \int \cos x . \frac{1}{\sin x} dx$$

$$= \int \cot x dx$$

$= \log(\sin x)$

$\Rightarrow \quad R = \log(\sin x)$ ... (7)

Substituting equations (6) and (7) in equation (5),

$y_p = -x(\cos x) + \log(\sin x) \sin x$

$\Rightarrow \quad y_p = -x\cos x + \log(\sin x). \sin x$ ... (8)

Substituting equations (4) and (8) in equation (3),

$y = c_1 \cos x + c_2 \sin x - x\cos x + \log(\sin x) \sin x + c$

$\therefore$ The general solution is,

$y = c_1 \cos x + c_2 \sin x - x\cos x + \sin x.\log(\sin x) + c$

Q42. Solve $\frac{d^2y}{dx^2} + y = \cot x$ by using method of variation of parameter.

Answer : Model Paper-1, Q13(b)

Given differential equation is,

$$\frac{d^2y}{dx^2} + y = \cot x$$

The above equation can be written as,

$(D^2 + 1)y = \cot x$ ... (1)

Equation (1) is a non-homogeneous linear differential equation of the form,

$[f(D)]y = X$ ... (2)

Comparing equations (1) and (2)

$f(D) = D^2 + 1$

$X = \cot x$

The general solution of a non-homogeneous linear differential equation is given as,

$y = \text{C.F} + \text{P.I}$ ... (3)

The auxiliary equation is given as,

$f(m) = 0$

i.e., $m^2 + 1 = 0$

$\Rightarrow \quad m^2 = -1$

$\Rightarrow \quad m^2 = i$

$\Rightarrow \quad m = \pm i$

The roots are complex conjugate of the form $\alpha \pm i\beta$

Here, $\alpha = 0, \beta = 1$

Hence, the complementary function is,

$\text{C.F} = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

$= e^{0x}(c_1 \cos x + c_2 \sin x)$

$= c_1 \cos x + c_2 \sin x$

$\therefore \quad \text{C.F} = c_1 \cos x + c_2 \sin x$ ... (4)

Here, $f_1 = \cos x$; $f_2 = \sin x$

$f_1' = -\sin x$; $f_2' = \cos x$

$f_1 f_2' - f_2 f_1' = (\cos x)(\cos x) - (\sin x)(-\sin x)$

$= \cos^2 x + \sin^2 x$

$\therefore f_1 f_2' - f_2 f_1' = 1$

The particular integral is given as,

$\text{P.I} = Pf_1 + Qf_2$ ... (5)

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_1' f_2} dx$$

$$= -\int \frac{\sin x \cot x}{1} dx$$

$$= -\int \sin x . \left(\frac{\cos x}{\sin x}\right) dx$$

$$= -\int \cos x \, dx$$

$$= -\sin x$$

$$\therefore \quad P = -\sin x \quad \text{... (6)}$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\cos x . \cot x}{1} dx$$

$$= \int \cos x \left(\frac{\cos x}{\sin x}\right) dx$$

$$= \int \frac{\cos^2 x}{\sin x} dx$$

$$= \int \frac{(1 - \sin^2 x)}{\sin x} dx$$

$$= \int \left(\frac{1}{\sin x} - \frac{\sin^2 x}{\sin x}\right) dx$$

$$= \int (\text{cosec}\, x - \sin x) dx$$

$$= \int \text{cosec}\, x dx - \int \sin x dx$$

$$= \log[\text{cosec}\, x - \cot x] + \cos x$$

$$\therefore \quad Q = \log[\text{cosec}\, x - \cot x] + \cos x \quad \text{... (7)}$$

Substituting equations (6) and (7) in equation (5),

$$\text{P.I} = -\sin x(\cos x) + [\log[\text{cosec}\, x - \cot x] + \cos x]\sin x$$

$$= -\sin x \cos x + \sin x . \log[\text{cosec}\, x - \cot x] + \sin x \cos x$$

$$= \log[\text{cosec}\, x - \cot x] \sin x$$

$$\therefore \quad \text{P.I} = \log[\text{cosec}\, x - \cot x] \sin x \quad \text{... (8)}$$

Substituting equations (4) and (8) in equation (3),

$$\therefore \quad y = c_1 \cos x + c_2 \sin x + \log[\text{cosec}\, x - \cot x]\sin x + c$$

Q43. Solve (D² + 2D + 5) y = e⁻ˣ tanx.

Answer :

Given differential equation is,

$$(D^2 + 2D + 5)\, y = e^{-x} \tan x \quad \text{... (1)}$$

Equation (1) is a non-homogeneous linear differential equation of the form

$$[f(D)]y = Q(x) \quad \text{... (2)}$$

Comparing equations (1) and (2),

$$f(D) = D^2 + 2D + 5$$

$$Q(x) = e^{-x} \tan x$$

The general solution is given as,

$$y = y_c + y_p \quad \text{... (3)}$$

The auxiliary equation is given as,

$$f(m) = 0$$

$\Rightarrow$ i.e., $m^2 + 2m + 5 = 0$

$$\Rightarrow \quad m = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$= \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2}$$

$$m = -1 \pm 2i$$

The roots are complex conjugate of the form $\alpha \pm i\beta$

The complementary function is given as,

$$y_c = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

$$\Rightarrow \quad y_c = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$$

$$y_c = c_1 e^{-x} \cos 2x + c_2 e^{-x} \sin 2x \quad \text{... (4)}$$

The particular integral is given by,

$$y_p = Pf_1 + Rf_2 \quad \text{... (5)}$$

Where, $f_1 = e^{-x} \cos 2x$

$$f_1' = e^{-x}(-2\sin 2x) + \cos 2x(-e^{-x})$$

$$\Rightarrow \quad f_1' = -2e^{-x} \sin 2x - e^{-x} \cos 2x$$

$$f_2 = e^{-x} \sin 2x$$

$$\Rightarrow \quad f_2' = e^{-x}(2\cos 2x) + \sin 2x(-e^{-x})$$

$$\Rightarrow \quad f_2' = 2e^{-x} \cos 2x - e^{-x} \sin 2x$$

$$f_1 f_2' - f_1' f_2 = e^{-x} \cos 2x\,(2e^{-x} \cos 2x - e^{-x} \sin 2x) - (-2e^{-x} \sin 2x - e^{-x} \cos 2x)\, e^{-x} \sin 2x$$

$$= 2e^{-2x} \cos^2 2x - e^{-2x} \sin 2x \cos 2x + 2e^{-2x} \sin^2 2x + e^{-2x} \sin 2x \cos 2x$$

$$= 2e^{-2x} \cos^2 2x + 2e^{-2x} \sin^2 2x$$

$$= 2e^{-2x}(1)$$

$$= 2e^{-2x}$$

$$P = -\int \frac{f_1 Q(x)}{f_1 f_2' - f_1' f_2} dx$$

$$= -\int \frac{e^{-x} \sin 2x (e^{-x} \tan x)}{2e^{-2x}} dx$$

$$= -\frac{1}{2} \int \frac{e^{-2x} \sin 2x . \tan x}{e^{-2x}} dx$$

$$= -\frac{1}{2} \int \sin 2x . \tan x \, dx$$

$$= -\frac{1}{2} \int (2 \sin x \cos x) . \frac{\sin x}{\cos x} dx$$

$$= -\int \sin^2 x \, dx$$

$$= -\int \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= -\frac{1}{2}\int (1-\cos 2x)\, dx$$

$$= \frac{-1}{2}\left[\int dx - \int \cos 2x\, dx\right]$$

$$\therefore \quad P = \frac{-1}{2}\left[x - \frac{\sin 2x}{2}\right] \qquad \text{... (6)}$$

$$R = \int \frac{f_1 Q(x)}{f_1 f_2' - f_1' f_2}\, dx$$

$$= \int \frac{e^{-x}\cos 2x\,(e^{-x}\tan x)}{2e^{-2x}}\, dx$$

$$= \frac{1}{2}\int \frac{e^{-2x}\cos 2x \tan x}{e^{-2x}}\, dx$$

$$= \frac{1}{2}\int \cos 2x.\tan x\, dx$$

$$= \frac{1}{2}\int (2\cos^2 x - 1)\frac{\sin x}{\cos x}\, dx$$

$$= \frac{1}{2}\int \left(2\cos^2 x\frac{\sin x}{\cos x} - \frac{\sin x}{\cos x}\right) dx$$

$$= \frac{1}{2}\int (2\sin x\cos x)\, dx - \frac{1}{2}\int \frac{\sin x}{\cos x}\, dx$$

$$= \frac{1}{2}\int \sin 2x\, dx + \frac{1}{2}\int \frac{(-\sin x)}{\cos x}\, dx$$

$$= \frac{1}{2}\left[\frac{-\cos 2x}{2}\right] + \frac{1}{2}\log(\cos x)$$

$$\therefore \quad R = \frac{-\cos 2x}{4} + \frac{1}{2}\log(\cos x) \qquad \text{... (7)}$$

Substituting equations (6) and (7) in equation (5)

$$y_p = \left[\frac{-x}{2} + \frac{\sin 2x}{4}\right]e^{-x}\cos 2x + \left[\frac{1}{2}\log(\cos x) - \frac{\cos 2x}{4}\right]e^{-x}\sin 2x$$

$$= -\frac{x}{2}e^{-x}\cos 2x + \frac{e^{-x}\sin 2x\cos 2x}{4} + \frac{1}{2}e^{-x}\sin 2x\,\log(\cos x) - \frac{e^{-x}\sin 2x\cos 2x}{4}$$

$$= \frac{-x}{2}e^{-x}\cos 2x + \frac{1}{2}e^{-x}\sin 2x\log(\cos x)$$

$$\therefore \; y_p = \frac{e^{-x}}{2}[\sin 2x\,\log(\cos x) - x\cos 2x] \qquad \text{... (8)}$$

Substituting equations (4) and (8) in equation (3),

$$y = c_1 e^{-x}\cos 2x + c_2\, e^{-x}\sin 2x + \frac{e^{-x}}{2}[\sin 2x\,\log(\cos x) - x\cos 2x] + c.$$

Q44. Write the solution of nth order Cauchy-Euler equation.

Answer :

The n^{th} order Cauchy's-Euler equation is given by,

$$x^n\frac{d^n y}{dx^n} + a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \text{..........} + a_{n-1}x\frac{dy}{dx} + a_n y = Q(x) \qquad \text{... (1)}$$

Where, a_0, a_1, ... a_{n-1}, a_n are constants.

Consider,

$$x = e^z \qquad \text{... (2)}$$

Applying logarithm on both sides,

$\log x = \log e^z$

$\Rightarrow \quad \log x = z \log e$

$\Rightarrow \quad \log x = z$

From equation (2),

$x = e^z$

$\Rightarrow \quad dx = e^z dz$

$\Rightarrow \quad \frac{dz}{dx} = \frac{1}{e^z}$

$\Rightarrow \quad \frac{dz}{dx} = \frac{1}{x}$ $\quad [\because$ From equation (2)$]$ ... (3)

Consider,

$$\frac{dy}{dx} = \frac{dy}{dx} \cdot \frac{dz}{dz}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dz}{dx} \cdot \frac{dy}{dz}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x}\left(\frac{dy}{dz}\right) \quad [\because \text{ From equation (3)}]$$

$$\Rightarrow \quad x\frac{dy}{dx} = \frac{dy}{dz} \quad \text{... (4)}$$

$$\Rightarrow \quad x\frac{dy}{dx} = D'y \quad \text{... (5)}$$

Where, $D' \equiv \frac{d}{dz}$

Differentiating equation (4) with respect to 'x',

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{d}{dx}\left(\frac{dy}{dz}\right)$$

Multiplying both sides by x,

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = x\frac{d}{dz}\left(\frac{dy}{dx}\right)$$

$$= x\frac{d}{dz}\left[\frac{1}{x}\frac{dy}{dz}\right] \quad [\because \text{ From equation (4)}]$$

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \frac{d^2y}{dz^2}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + \frac{dy}{dz} = \frac{d^2y}{dz^2} \quad [\because \text{ From equation (4)}]$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} = (D'^2 - D')y$$

Where, $D' \equiv \frac{d}{dz}$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} = D'(D'-1)y \quad \text{... (6)}$$

Similarly

$$\Rightarrow \quad x^3\frac{d^3y}{dx^3} = D'(D'-1)(D'-2)y \quad \text{... (7)}$$

$$\Rightarrow \quad x^4\frac{d^4y}{dx^4} = D'(D'-1)(D'-2)(D'-3)y \quad \text{... (8)}$$

Substituting equations (5), (6), (7) and (8) in equation (1) to obtain a linear differential equation with constant coefficients.

Q45. Solve $(x^2D^2 - 3xD)\, y = x + 11$.

Answer : **Model Paper-1, Q16(b)**

Given differential equation is,

$(x^2D^2 - 3xD)\, y = x + 11$ ... (1)

Let $x = e^z$

$\Rightarrow \quad z = \log x$

Then $xD \equiv D'$

$x^2D^2 \equiv D'(D'-1)$

Where, $D' \equiv \frac{d}{dz}$

Equation (1) becomes,

$(D'(D'-1) - 3D')\, y = e^z + 11$

$\Rightarrow \quad ((D')^2 - D' - 3D')\, y = e^z + 11$

$\Rightarrow \quad [(D')^2 - 4D']\, y = e^z + 11$ ... (2)

Equation (2) is a non-homogeneous linear differential equation of the form,

$[f(D)]\, y = X$ ... (3)

Comparing equation (2) and (3),

$f(D) = (D')^2 - 4D'$

$X = e^z + 11$

The general solution is given as,

$y = \text{C.F} + \text{P. I}$ ... (4)

The auxiliary equation is given as,

$f(m) = 0$

$\Rightarrow \quad m^2 - 4m = 0$

$\Rightarrow \quad m(m-4) = 0$

$\Rightarrow \quad m = 0, m = 4$

The roots are real and distinct.

The complementary function is given as,

$\text{C.F} = c_1 e^{m_1 z} + c_2 e^{m_2 z}$

$\Rightarrow \quad \text{C.F} = c_1 e^{0.z} + c_2 e^{4z}$

$\therefore \quad \text{C.F} = c_1 + c_2 e^{4z}$ ... (5)

The particular integral is given as,

$$\text{P.I} = \frac{X}{f(D)} = \frac{e^z + 11}{(D')^2 - 4D'}$$

$$\text{P.I} = \frac{e^z}{(D')^2 - 4D'} + \frac{11.e^{0.z}}{(D')^2 - 4D'} \quad \text{... (6)}$$

Consider, $\frac{e^z}{(D')^2 - 4D'}$

Here X is of the form e^{ax}

Let, $D = a$ i.e., $D = 1$

$$\frac{e^z}{(D')^2 - 4D'} = \frac{e^z}{(1)^2 - 4(1)}$$

$$= \frac{e^z}{1-4}$$

$$\frac{e^z}{(D')^2 - 4D'} = \frac{e^z}{-3} \quad \text{... (7)}$$

Consider, $\frac{11e^{0.z}}{(D')^2 - 4D'}$

Here X is of the form e^{ax},

Let, $D = a$ i.e., $D = 0$

$$\frac{11e^{0.z}}{(D')^2 - 4D'} = \frac{z}{2D'-4} 11.e^{0.z} \quad \left[\begin{array}{c} \because f(D) \text{ at } D = 0 \\ \Rightarrow f(D) = 0 \\ P.I = \frac{X}{f'(D)} e^{ax} \end{array}\right]$$

Let $D = a$ i.e., $D = 0$

$$= \frac{z.(11.e^{0.z})}{2(0)-4}$$

$$= \frac{-11}{4} ze^{0.z}$$

$$\frac{11.e^{0.z}}{(D')^2 - 4D'} = \frac{-11}{4} z \quad \text{... (8)}$$

Substituting equations (7) and (8) in equation (6),

$$\text{P.I} = -\frac{e^z}{3} - \frac{11}{4} z \quad \text{... (9)}$$

Substituting equations (5) and (9) in equation(4),

$$y = c_1 + c_2 e^{4z} - \frac{e^z}{3} - \frac{11z}{4}$$

$$\Rightarrow \quad y = c_1 + c_2 e^{4\log x} - \frac{x}{3} - \frac{11 \log x}{4}$$

$$\Rightarrow \quad y = c_1 + c_2 x^4 - \frac{x}{3} - \frac{11 \log x}{4}$$

$\therefore$ The general solution is,

$$y = c_1 + c_2 x^4 - \frac{x}{3} - \frac{11}{4} \log x$$

Q46. Solve $x^2y'' - xy' + y = \log x + \pi$.

Answer :

Given differential equation is,

$$x^2 y'' - xy' + y = \log x + \pi$$

The above equation can be written as,

$$(x^2D^2 - xD + 1)\, y = \log x + \pi \quad \text{... (1)}$$

Let $x = e^z \Rightarrow z = \log x$

Then $x D \equiv D'$

$$x^2D^2 \equiv D'(D' - 1)$$

Where $D' \equiv \frac{d}{dz}$

Equation (1) becomes

$$\Rightarrow \quad (D'(D' - 1) - D' + 1)\, y = z + \pi$$

$$\Rightarrow \quad [(D')^2 - D' - D' + 1]\, y = z + \pi$$

$$\Rightarrow \quad [(D')^2 - 2D' + 1]\, y = z + \pi \quad \text{... (2)}$$

Equation (2) is a non-homogeneous linear differential equation of the form,

$$[(f(D)]\, y = X \quad \text{... (3)}$$

Comparing equations (2) and (3)

$$f(D) = (D')^2 - 2D' + 1$$

$$X = z + \pi$$

The general solution is given as,

$$y = \text{C.F} + \text{P. I} \quad \text{... (4)}$$

The auxiliary equation is given as,

$$f(m) = 0$$

$$\Rightarrow \quad m^2 - 2m + 1 = 0$$

$$\Rightarrow \quad (m - 1)(m - 1) = 0$$

$$\Rightarrow \quad m = 1, 1$$

The roots are real and equal

The complementary function is given as,

$$\text{C. F} = (c_1 + c_2 z)\, e^z \quad \text{... (5)}$$

The particular integral is given as,

$$\text{P.I} = \frac{X}{f(D)}$$

$$= \frac{z + \pi}{(D')^2 - 2D' + 1}$$

$$= \frac{1}{1 + [(D')^2 - 2D']}(z + \pi)$$

$$= \left[1 + [D')^2 - 2D']\right]^{-1}(z + \pi)$$

$$= [1 - [(D')^2 - 2D'] + ...](z + \pi)$$

$$= [1 + 2D'](z + \pi)$$

$$= z + \pi + 2D'(z + \pi)$$

$$= z + \pi + 2(1 + 0)$$

$$\therefore \quad \text{P. I} = z + \pi + 2 \quad \text{... (6)}$$

Substituting equations (5) and (6) in equation (4),

$$y = (c_1 + c_2 z)\, e^z + z + \pi + 2$$

$$\Rightarrow \quad y = (c_1 + c_2 \log x)\, x + \log x + \pi + 2$$

$\therefore$ The general solution is,

$$y = (c_1 + c_2 \log x)\, x + \log x + \pi + 2.$$

Q47. Solve the equation $x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \sin(\log x)$.

Answer : **Model Paper-3, Q16(b)**

Given differential equation is,

$$x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = \sin(\log x)$$

The above equation can be written as,

$$(x^2 D^2 + 4x D + 2)\, y = \sin(\log x) \qquad \ldots (1)$$

Let, $x = e^z \quad \Rightarrow z = \log x$

Then, $xD \equiv D'$

$$x^2D^2 \equiv D'(D' - 1)$$

Where, $D' \equiv \frac{d}{dz}$

Equation (1) becomes,

$$[D'(D' - 1) + 4D' + 2]\, y = \sin z$$

$\Rightarrow \quad [(D')^2 - D' + 4D' + 2]\, y = \sin z$

$\Rightarrow \quad [(D')^2 + 3D' + 2]\, y = \sin z \qquad \ldots (2)$

Equation (2) is a non-homogeneous linear differential equation of the form,

$$[f(D)] = X \qquad \ldots (3)$$

Comparing equations (2) and (3),

$$f(D) = (D')^2 + 3D' + 2$$

$$X = \sin z$$

The general solution is given as,

$$y = \text{C. F} + \text{P. I} \qquad \ldots (4)$$

The auxiliary equation is given as,

$$f(m) = 0$$

$\Rightarrow \quad m^2 + 3m + 2 = 0$

$\Rightarrow \quad m^2 + m + 2m + 2 = 0$

$\Rightarrow \quad m(m + 1) + 2(m + 1) = 0$

$\Rightarrow \quad (m + 1)(m + 2) = 0$

$\Rightarrow \quad m = -1, -2$

The roots are real and distinct

The complementary function is given as,

$$\text{C. F} = c_1 e^{-z} + c_2 e^{-2z} \qquad \ldots (5)$$

The particular integral is given as,

$$\text{P. I} = \frac{X}{f(D)}$$

$$= \frac{\sin z}{(D')^2 + 3D' + 2}$$

Here X is in the form $\sin ax$

Let $(D')^2 = -a^2$ i.e., $(D')^2 = -1$

$$\text{P. I} = \frac{\sin z}{-1+3D'+2}$$

$$= \frac{\sin z}{3D'+1}$$

$$= \left[\frac{1}{3D'+1} \times \frac{3D'-1}{3D'-1}\right]\sin z$$

$$= \left[\frac{3D'-1}{(3D')^2-(1)^2}\right]\sin z$$

$$= \left[\frac{3D'-1}{9(D')^2-1}\right]\sin z$$

Let $(D')^2 = -1$

$$= \frac{3D'-1}{9(-1)-1}\sin z$$

$$= \frac{3D'-1}{-10}\sin z$$

$$= \frac{-1}{10}[3D'(\sin z) - \sin z]$$

$$\text{P. I} = \frac{-1}{10}[3\cos z - \sin z] \qquad \text{... (6)}$$

Substituting equations (5) and (6) in equation (4),

$$y = c_1 e^{-z} + c_2 e^{-2z} - \frac{1}{10}[3\cos z - \sin z]$$

$$\Rightarrow \quad y = c_1 \frac{1}{x} + c_2 \frac{1}{x^2} - \frac{1}{10}[3\cos(\log x) - \sin(\log x)]$$

$$\Rightarrow \quad y = c_1 x^{-1} + c_2 x^{-2} - \frac{[3\cos(\log x) - \sin(\log x)]}{10}$$

$\therefore$ The general solution is,

$$y = c_1 x^{-1} + c_2 x^{-2} - \frac{1}{10}[3\cos(\log x) - \sin(\log x)].$$

Q48. Solve $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = \log x \sin(\log x)$.

Answer : Model Paper-2, Q13(b)

Given differential equation is,

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x \sin(\log x)$$

The above equation can be written as,

$$[x^2D^2 + xD + 1]\, y = \log x \sin(\log x) \qquad \text{... (1)}$$

Let, $x = e^z \Rightarrow z = \log x$

Then $xD \equiv D'$

$$x^2D^2 \equiv D'(D'-1)$$

Where $D' \equiv \dfrac{d}{dz}$

Equation (1) becomes

$$(D'(D'-1) + D' + 1]\, y = z \sin z \qquad \text{... (2)}$$

Equation (2) is a non-homogeneous linear differential equation of the form

$$[f(D)]\, y = x \qquad \text{... (3)}$$

Comparing equations (2) and (3)

$f(D) = D'(D'-1) + D' + 1$

$= (D')^2 - D' + D' + 1$

$= (D')^2 + 1$

$X = z \sin z$

The general solution is given as,

$y = \text{C.F} + \text{P. I}$ ... (4)

The auxiliary equation is given as,

$f(m) = 0$

$\Rightarrow \quad m^2 + 1 = 0$

$\Rightarrow \quad m^2 = -1$

$\Rightarrow \quad m = \pm i$

Since the roots are complex conjugate of the form $\alpha \pm i\beta$

The complementary function is given as,

$\text{C.F} = e^{0.z}(c_1 \cos z + c_2 \sin z)$

$\Rightarrow \quad \text{C.F} = c_1 \cos z + c_2 \sin z$ ... (5)

The particular integral is given as,

$$\text{P.I} = \frac{X}{f(D)}$$

$$\Rightarrow \quad \text{P.I} = \frac{1}{(D')^2+1}(z \sin z)$$

$$= \text{Imaginary part of } \frac{1}{(D')^2+1}(ze^{iz})$$

$$= \text{I.P of } e^{iz} \frac{1}{(D')^2+1} z$$

Let, $D' = D' + i$

$$P.I = \text{I.P of } e^{iz} \frac{1}{(D'+i)^2+1} z$$

$$= \text{I.P of } e^{iz} \frac{1}{(D')^2+(i)^2+2D'i+1} z$$

$$= \text{I.P of } e^{iz}. \frac{1}{(D')^2-1+2D'i+1} z$$

$$= \text{I.P of } e^{iz} \frac{1}{(D')^2+2D'i}.z$$

$$= \text{I.P of } e^{iz} \frac{1}{2D'i\left[1+\frac{(D')^2}{2D'i}\right]}.z$$

$$= \text{I.P of } e^{iz} \frac{1}{2D'i}\cdot\left[\frac{1}{1+\frac{D'}{2i}}\right] z$$

$$= \text{I.P of } e^{iz} \frac{1}{2D'i}\left[1+\frac{D'}{2i}\right]^{-1} z$$

$$= \text{I.P of } e^{iz} \cdot \frac{1}{2D'i} \cdot \left[1 - \frac{D'}{2i}\right] . z$$

$$= \text{I.P of } e^{iz} \cdot \frac{1}{2D'i} \cdot \left[z - \frac{D'(z)}{2i}\right]$$

$$= \text{I.P of } e^{iz} \cdot \frac{1}{2D'i} \cdot \left[z - \frac{1}{2i}\right]$$

$$= \text{I.P of } e^{iz} \cdot \frac{1}{2i}\left[\frac{1}{D'}(z) - \frac{1}{2i}\frac{1}{D'}(1)\right]$$

$$= \text{I.P of } e^{iz} \cdot \frac{1}{2i}\left[\frac{z^2}{2} - \frac{1}{2i}(z)\right] = \text{I.P of } e^{iz} \cdot \left[\frac{z^2}{4i} - \frac{z}{4i^2}\right]$$

$$= \text{I.P of } (\cos z + i \sin z) \left[\frac{z^2}{4}\left(\frac{1}{i}\right) - \frac{z}{4(-1)}\right]$$

$$= \text{I.P of } (\cos z + i \sin z)\left[\frac{z^2}{4}(-i) + \frac{z}{4}\right]$$

$$= \text{I.P of } (\cos z + i \sin z)\left[-i\frac{z^2}{4} + \frac{z}{4}\right]$$

$$= \text{I.P of } \left[-i\frac{z^2}{4}\cos z + \frac{z}{4}\cos z - i^2 \sin z\left(\frac{z^2}{4}\right) + i \sin z\left(\frac{z}{4}\right)\right]$$

$$= \text{I.P of } \left[-i\frac{z^2}{4}\cos z + \frac{z}{4}\cos z + \frac{z^2}{4}\sin z + i\frac{z}{4}\sin z\right]$$

$$\text{P. I} = -\frac{z^2}{4}\cos z + \frac{z}{4}\sin z \qquad \dots (6)$$

Substituting equations (5) and (6) in equation (4),

$$y = c_1 \cos z + c_2 \sin z - \frac{z^2}{4}\cos z + \frac{z}{4}\sin z$$

$$\Rightarrow \quad y = c_1 \cos(\log x) + c_2 \sin(\log x) - \frac{(\log x)^2}{4}\cos(\log x) + \frac{\log x}{4}\sin(\log x)$$

$\therefore$ The general solution is,

$$y = c_1 \cos(\log x) + c_2 \sin(\log x) - \frac{(\log x)^2}{4}\cos(\log x) + \frac{\log x}{4}\sin(\log x)$$

UNIT 4

SPECIAL FUNCTION

PART-A

SHORT QUESTIONS WITH SOLUTIONS

Q1. Define Gamma function and list its important formulae.

Answer: Model Paper-1, Q7

Gamma Function

The definite integral $\int_0^\infty e^{-x}.x^{n-1}.dx$ (for $n > 0$), is termed as Gamma function. It is a function of 'n' and is denoted by 'Γ'.

$$\Gamma(n) = \int_0^\infty e^{-x}.x^{n-1}.dx \qquad (n > 0)$$

It is also known as 'Euler's integral of the second kind'.

Formulae

(i) $\Gamma(n + 1) = n\Gamma(n)$ or $\Gamma(n) = (n - 1)\,\Gamma(n - 1)$

(ii) $\Gamma(n) = (n - 1)!$ or $\Gamma(n + 1) = n!$

(iii) $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$

(iv) $\Gamma(n).\Gamma(1 - n) = \dfrac{\pi}{\sin n\pi}$

(v) $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$

Q2. Find the value of $\Gamma\left(\frac{7}{2}\right)$.

Answer: June-11, Q7

Given function is,

$$\Gamma\left(\frac{7}{2}\right)$$

Gamma function interms of n is given by,

$$\Gamma(n) = (n - 1)\Gamma(n - 1)$$

Substituting $n = \dfrac{7}{2}$ in above equation,

$$\Gamma\left(\frac{7}{2}\right) = \left(\frac{7}{2}-1\right)\Gamma\left(\frac{7}{2}-1\right)$$

$$= \frac{5}{2}\Gamma\left(\frac{5}{2}\right)$$

$$= \frac{5}{2}\left[\left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right)\right]$$

$$= \frac{5}{2}\left[\left(\frac{3}{2}\Gamma\left(\frac{3}{2}\right)\right)\right]$$

$$= \frac{5}{2}\left[\left(\frac{3}{2}\left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right)\right)\right]$$

$$= \frac{5}{2}\left[\frac{3}{2}\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right]$$

$$= \frac{5}{2}\left[\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}\right] \qquad \left(\because \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\right)$$

$$= \frac{15\sqrt{\pi}}{8}$$

$$\therefore \quad \Gamma\left(\frac{7}{2}\right) = \frac{15\sqrt{\pi}}{8}$$

Q3. Find the value of $\Gamma\left(\frac{9}{2}\right)$.

Answer : Dec.-12, Q10

Given function is,

$$\Gamma\left(\frac{9}{2}\right)$$

The above function can be written as,

$$\Gamma\left(\frac{9}{2}\right) = \Gamma\left(\frac{7}{2}+1\right)$$

From the property of Gamma function,

$$\Gamma(n+1) = n\Gamma(n)$$

$$\therefore \quad \Gamma\left(\frac{9}{2}\right) = \Gamma\left(\frac{7}{2}+1\right) = \frac{7}{2}\times\Gamma\left(\frac{7}{2}\right)$$

$$\Rightarrow \quad \Gamma\left(\frac{9}{2}\right) = \frac{7}{2}\Gamma\left(\frac{7}{2}\right)$$

$$= \frac{7}{2}\Gamma\left(\frac{5}{2}+1\right)$$

$$= \frac{7}{2}\times\frac{5}{2}\times\Gamma\left(\frac{5}{2}\right)$$

$$= \frac{7}{2}\times\frac{5}{2}\times\Gamma\left(\frac{3}{2}+1\right)$$

$$= \frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\Gamma\left(\frac{3}{2}\right)$$

$$= \frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\Gamma\left(\frac{1}{2}+1\right)$$

$$= \frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi} \qquad \left(\because \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}\right)$$

$$= \frac{105\sqrt{\pi}}{16}$$

$$\therefore \quad \Gamma\left(\frac{9}{2}\right) = \frac{105\sqrt{\pi}}{16}$$

Q4. Show that $\int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy = \Gamma(n)$

Answer : Model Paper-2, Q7

Given integral is,

$$\int_0^1 (\log_e 1/y)^{n-1}\, dy$$

Let, $\log_e \frac{1}{y} = t$

$$\Rightarrow \quad \frac{1}{y} = e^t \quad \Rightarrow \quad y = e^{-t}$$

Differentiating on both sides with respect to 't',

$$dy = -e^{-t}\, dt$$

Limits

For $y = 0$, $\log_e \frac{1}{0} = t \Rightarrow \log_e \infty = t \Rightarrow t = \infty$

For $y = 1$, $\log_e \frac{1}{1} = t \Rightarrow \log_e 1 = t \Rightarrow t = 0$

$\therefore$ Limits are from ∞ to 0.

Then,

$$\int_0^1 \left(\log_e \frac{1}{y}\right)^{n-1} dy = \int_\infty^0 (t)^{n-1}.(-e^{-t}).dt$$

$$= -\int_\infty^0 e^{-t}.t^{n-1}.dt$$

$$= \int_0^\infty e^{-t}.t^{n-1}.dt$$

$$= \Gamma(n) \qquad \left(\because \int_0^\infty e^{-x}.x^{n-1}.dx = \Gamma(n)\right)$$

$$\therefore \quad \Gamma(n) = \int_0^1 \left(\log_e \frac{1}{y}\right)^{n-1} dy$$

Q5. Prove that $\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

Answer : Dec.-13, Q8

Given that,

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

The general expression of a β-function in terms of (m, n) is,

$$\beta(m, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}.dx \qquad \text{... (1)}$$

Relation between 'Γ' and 'β' function is given by,

$$\beta(m, n) = \frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)} \qquad \text{... (2)}$$

Combining equations (1) and (2),

$$\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad \text{... (3)}$$

Let, $m + n = 1 \Rightarrow m = 1 - n$

$\therefore$ Equation (3) becomes,

$$\int_0^\infty \frac{x^{n-1}}{(1+x)^1}dx = \frac{\Gamma(1-n)\Gamma(n)}{\Gamma(1)}$$

$$= \frac{\Gamma(1-n)\Gamma(n)}{1} \quad [\because \Gamma(1) = 1]$$

$$\therefore \int_0^\infty \frac{x^{n-1}}{(1+x)}.dx = \Gamma(1-n)\ \Gamma(n) \quad \text{... (4)}$$

Also,

$$\int_0^\infty \frac{x^{2m}}{(1+x^{2n})}dx = \frac{\pi}{2n}\text{cosec}\frac{(2m+1)\pi}{2n} \quad \text{... (5)}$$

Where, $m > 0$, $n > 0$ and $n > m$

Let $x^{2n} = t \Rightarrow x = (t)^{1/2n}$ and $\frac{(2m+1)}{2n} = p$

Differentiating with respect to 'x',

$$dx = \frac{1}{2n}t^{\frac{1}{2n}-1}.dt$$

Limits

For $x = 0$, $0 = (t^{1/2n}) \Rightarrow t = 0$

For $x = \infty$, $\infty = (t^{1/2n}) \Rightarrow t = \infty$

$\therefore$ Limits are from 0 to ∞.

Substituting the corresponding values in equation (5),

$$\int_0^\infty \frac{(t^{1/2n})^{2m}}{(1+t)}.\frac{1}{2n}t^{\frac{1}{2n}-1}dt = \frac{\pi}{2n}\text{cosec}\,p\pi$$

$$\Rightarrow \int_0^\infty \frac{t^{\frac{2m}{2n}}}{(1+t)}.\frac{1}{2n}.\frac{t^{\frac{1}{2n}-1}}{t}dt = \frac{\pi}{2n}\text{cosec}\,p\pi$$

$$\Rightarrow \int_0^\infty \frac{t^{(2m/2n)}}{2n(1+t)}.t^{\frac{1}{2n}-1}dt = \frac{\pi}{2n}\text{cosec}\,p\pi$$

$$\Rightarrow \frac{1}{2n}\int_0^\infty \frac{t^{\frac{2m}{2n}}.t^{\frac{1}{2n}-1}}{(1+t)}dt = \frac{\pi}{2n}\text{cosec}\,p\pi$$

$$\Rightarrow \int_0^\infty \frac{t^{\left(\frac{2m+1}{2n}\right)-1}}{1+t}dt = \pi\,\text{cosec}\,p\pi \quad \left(\because p = \frac{2m+1}{2n}\right)$$

$$\Rightarrow \int_0^\infty \frac{t^{p-1}}{1+t}dt = \frac{\pi}{\sin p\pi} \quad \text{... (6)}$$

Equation (6) can also be expressed interms of 'x' as,

$$\int_0^\infty \frac{x^{p-1}}{1+x}dx = \frac{\pi}{\sin p\pi}$$

Let $p = n$

$$\therefore \int_0^\infty \frac{x^{n-1}}{(1+x)}dx = \frac{\pi}{\sin n\pi} \quad \text{... (7)}$$

Combining equations (4) and (7),

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

$$\therefore \quad \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

Q6. Evaluate $\int_0^\infty x^2 e^{-x^2} dx$.

Answer : **(Model Paper-1, Q8 | June-13, Q9 | June-10, Q9)**

Given integral is,

$$\int_0^\infty x^2.e^{-x^2}dx$$

Let, $x^2 = t$

$\Rightarrow \quad x = \sqrt{t}$

$$dx = \frac{1}{2\sqrt{t}}dt$$

$$\therefore \int_0^\infty x^2.e^{-x^2}dx = \int_0^\infty t.e^{-t}\left(\frac{1}{2\sqrt{t}}\right)dt$$

$$= \frac{1}{2}\int_0^\infty \frac{t}{\sqrt{t}}.e^{-t}.dt = \frac{1}{2}\int_0^\infty t^{1/2}.e^{-t}dt$$

$$= \frac{1}{2}\int_0^\infty t^{\frac{3}{2}-1}.e^{-t}dt$$

$$= \frac{1}{2}\times\Gamma\left(\frac{3}{2}\right) \quad \left(\because \int_0^\infty x^{n-1}.e^{-x}dx = \Gamma(n)\right)$$

$$= \frac{1}{2}\left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right) \quad [\because \Gamma(n) = (n-1)\,\Gamma(n-1)]$$

$$= \frac{1}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{4}\sqrt{\pi} \quad \left(\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$$

$$= \frac{\sqrt{\pi}}{4}$$

$$\therefore \int_0^\infty x^2 e^{-x^2}dx = \frac{\sqrt{\pi}}{4}$$

Q7. Evaluate $\int_0^1 \frac{x}{\sqrt{1-x^2}}dx$.

Answer : **Dec.-12, Q9**

Given integral is,

$$\int_0^1 \frac{x}{\sqrt{1-x^2}}dx$$

Let, $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

Limits

For $x = 0 \Rightarrow \theta = 0$

For $x = 1 \Rightarrow \theta = \frac{\pi}{2}$

$\therefore$ Limits are from 0 to $\frac{\pi}{2}$

Then,

$$\int_0^1 \frac{x}{\sqrt{1-x^2}}dx = \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \times \cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\cos\theta}\cos\theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} (\sin\theta)^1 d\theta$$

$$\left[\because \int_0^{\frac{\pi}{2}} (\sin\theta)^n d\theta = \frac{1}{2}\frac{\sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}\right]$$

$$= \frac{\sqrt{\pi}\Gamma\left(\frac{1+1}{2}\right)}{2\Gamma\left(\frac{1+2}{2}\right)} = \frac{\sqrt{\pi}\Gamma\left(\frac{2}{2}\right)}{2\Gamma\left(\frac{1}{2}+1\right)}$$

$$= \frac{\sqrt{\pi}.\Gamma(1)}{2\Gamma\left(\frac{1}{2}+1\right)}$$

$$= \frac{\sqrt{\pi}.(1)}{2.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} \qquad \left[\begin{array}{l}\because \Gamma(1) = 1 \\ \Gamma(n+1) = n\Gamma(n)\end{array}\right]$$

$$= \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1 \qquad \left(\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$$

$$\therefore \int_0^1 \frac{x}{\sqrt{1-x^2}}dx = 1$$

Q8. Evaluate $\int_0^{\infty} x^{\frac{1}{3}} e^{-x^2} dx$.

Answer : Dec.-17, Q7

Given integral is,

$$\int_0^{\infty} x^{\frac{1}{3}} e^{-x^2} dx$$

Let $x^2 = y \Rightarrow x = \sqrt{y}$

$2xdx = dy$

$dx = \frac{dy}{2x}$

$dx = \frac{1}{2\sqrt{y}}dy$

U.L : If $x = \infty$, $y = \infty$

L.L : If $x = 0$, $y = 0$

Then,

$$\int_0^{\infty} x^{\frac{1}{3}} e^{-x^2} dx = \int_0^{\infty} (\sqrt{y})^{\frac{1}{3}} e^{-y} \frac{1}{2\sqrt{y}} dy$$

$$= \int_0^{\infty} y^{\frac{1}{2}\times\frac{1}{3}} e^{-y} \frac{1}{2} y^{\frac{-1}{2}} dy$$

$$= \frac{1}{2}\int_0^{\infty} e^{-y} y^{\frac{1}{6}-\frac{1}{2}} dy$$

$$= \frac{1}{2}\int_0^{\infty} e^{-y} y^{-\frac{1}{3}} dy$$

$$= \frac{1}{2}\int_0^{\infty} e^{-y} y^{\frac{2}{3}-1} dy$$

$$= \frac{1}{2}\Gamma\left(\frac{2}{3}\right)$$

$$\therefore \int_0^{\infty} x^{\frac{1}{3}} e^{-x^2} dx = \frac{1}{2}\Gamma\left(\frac{2}{3}\right).$$

Q9. Evaluate $\int_0^{\infty} t^4.e^{-2t^2}.dt$.

Answer : May/June-17, Q7

Given integral is,

$$\int_0^{\infty} t^4 e^{-2t^2} dt$$

From the property of gamma function,

$$\int_0^{\infty} x^m e^{-ax^n} dx = \frac{1}{n a^{\frac{m+1}{n}}} \Gamma\left(\frac{1+m}{n}\right)$$

Here, $m = 4$, $a = 2$, $n = 2$

$$\int_0^{\infty} t^4 e^{-2t^2} dt = \frac{1}{2\, 2^{\frac{4+1}{2}}} \Gamma\left(\frac{1+4}{2}\right)$$

$$= \frac{1}{2.2^{\frac{5}{2}}}\Gamma\left(\frac{5}{2}\right)$$

$$= 2^{\frac{1}{5/2+1}}\left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right)$$

$$[\because \Gamma(n) = (n-1)\,\Gamma(n-1)]$$

$$= \frac{1}{2^{\frac{7}{2}}}\frac{3}{2}\Gamma\left(\frac{3}{2}\right)$$

$$= \frac{3}{2^{\frac{7}{2}+1}}\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{3}{2^{\frac{9}{2}+1}}\sqrt{\pi} \qquad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \frac{3}{2^{\frac{11}{2}}}\sqrt{\pi}$$

$$= \frac{3\sqrt{\pi}}{2^{5+\frac{1}{2}}}$$

$$= \frac{3\sqrt{\pi}}{32\sqrt{2}}$$

$$= \frac{3}{32}\sqrt{\frac{\pi}{2}}$$

$$\int_0^\infty t^4 e^{-2t^2} dt = \frac{3}{32}\sqrt{\frac{\pi}{2}}.$$

Q10. Define Beta function. State its formulae.

Answer : Model Paper-2, Q8

Beta Function

The definite integral $\int_0^1 x^{m-1}.(1-x)^{n-1}.dx$ $(m > 0, n > 0)$ is termed as Beta function. It is a function of m, n and is denoted by β.

$$\beta(m, n) = \int_0^1 x^{m-1}.(1-x)^{n-1} dx \; (m > 0, n > 0)$$

It is also known as 'Euler's integral of the first kind'.

Other Forms of β Function

❖ Trigonometric form :

$$\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta.\cos^{2n-1}\theta.d\theta$$

$$= 2.I_{2m-1,\, 2n-1}$$

❖ Beta function in terms of improper integral :

$$\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}.dx$$

Formulae

(i) $\beta(m, n) = \beta(n, m)$

(ii) $\beta(m, n) = \dfrac{(m-1)!(n-1)!}{(m+n-1)!}$

(iii) $\beta(m, n) = \beta(m + 1, n) + \beta(m, n + 1)$

(iv) $\beta(m,n) = \dfrac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}$

Q11. Show that β(m, n) = β(n, m).

Answer : Model Paper-3, Q7

Given that,

$$\beta(m, n) = \beta(n, m)$$

The general expression for Beta function in terms of m, n is,

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx \qquad \text{... (1)}$$

Let, $1 - x = t \Rightarrow x = (1 - t)$

Differentiating on both sides with respect to 'x',

$$dx = -dt$$

Limits

For $x = 0$, $1 - 0 = t \Rightarrow t = 1$

For $x = 1$, $1 - 1 = t \Rightarrow t = 0$

∴ Limits are from 1 to 0.

Substituting the corresponding values in equation (1),

$$\beta(m, n) = \int_1^0 (1-t)^{m-1} t^{n-1}(-dt)$$

$$= \int_0^1 t^{n-1}(1-t)^{m-1} dt$$

$$= \beta(n, m) \qquad [\because \text{From equation (1)}]$$

$$\therefore \quad \beta(m, n) = \beta(n, m)$$

Q12. Find the value of $\beta\left(\frac{9}{2}, \frac{7}{2}\right)$

Answer : Jan.-12, Q7

Given function is,

$$\beta\left(\frac{9}{2}, \frac{7}{2}\right)$$

The relation between 'β' and Γ function is,

$$\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Substituting $m = \frac{9}{2}$ and $n = \frac{7}{2}$ in above equation,

$$\beta\left(\frac{9}{2}, \frac{7}{2}\right) = \frac{\Gamma\left(\frac{9}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{9}{2}+\frac{7}{2}\right)}$$

$$= \frac{\left(\frac{9}{2}-1\right)\Gamma\left(\frac{9}{2}-1\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{16}{2}\right)} \qquad [\because \Gamma(n) = (n-1)\,\Gamma(n-1)]$$

$$= \frac{\frac{7}{2}\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma(8)}$$

$$= \frac{\frac{7}{2}\left[\Gamma\left(\frac{7}{2}\right)\right]^2}{\Gamma(8)}$$

$$= \frac{\frac{7}{2}\left[\left(\frac{7}{2}-1\right)\Gamma\left(\frac{7}{2}-1\right)\right]^2}{(8-1)!} \quad [\Gamma(n) = (n-1)!]$$

$$= \frac{\frac{7}{2}\left[\frac{5}{2}\Gamma\left(\frac{5}{2}\right)\right]^2}{7!}$$

$$= \frac{\frac{7}{2}\times\left(\frac{5}{2}\right)^2\left[\Gamma\left(\frac{5}{2}\right)\right]^2}{7!}$$

$$= \frac{\frac{7}{2}\times\frac{25}{4}\times\left[\left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right)\right]^2}{7\times6\times5\times4\times3\times2\times1}$$

$$= \frac{\frac{7}{2}\times\frac{25}{4}\left[\frac{3}{2}\Gamma\left(\frac{3}{2}\right)\right]^2}{5040}$$

$$= \frac{\frac{7}{2}\times\frac{25}{4}\times\frac{9}{4}\times\left[\left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right)\right]^2}{5040}$$

$$= \frac{\frac{7}{2}\times\frac{25}{4}\times\frac{9}{4}\times\left[\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right]^2}{5040}$$

$$= \frac{\frac{7}{2}\times\frac{25}{4}\times\frac{9}{4}\times\left(\frac{1}{2}\right)^2\left(\Gamma\left(\frac{1}{2}\right)\right)^2}{5040}$$

$$= \frac{\frac{7}{2}\times\frac{25}{4}\times\frac{9}{4}\times\frac{1}{4}\times\left(\sqrt{\pi}\right)^2}{5040} \quad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \frac{\frac{7}{2}\times\frac{25}{4}\times\frac{9}{4}\times\frac{1}{4}\times\pi}{5040}$$

$$= \frac{\left[\frac{7\times25\times9\times1\times\pi}{2\times4\times4\times4}\right]}{5040}$$

$$= \frac{\left[\frac{1575\pi}{128}\right]}{5040}$$

$$= \frac{5\pi}{2048}$$

$$\therefore \beta\left(\frac{9}{2},\frac{7}{2}\right) = \frac{5\pi}{2048}$$

Q13. Evaluate $\int_0^\infty \frac{xdx}{(1+x^6)}$ using β-Γ functions.

Answer :

Given integral is,

$$\int_0^\infty \frac{xdx}{1+x^6} \quad \text{... (1)}$$

Let, $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

Limits

For $x = 0$, $0 = \tan\theta \Rightarrow \tan 0 = \tan\theta \Rightarrow \theta = 0$

For $x = \infty$, $\infty = \tan\theta \Rightarrow \tan\frac{\pi}{2} = \tan\theta \Rightarrow \theta = \frac{\pi}{2}$

∴ Limits are from 0 to $\frac{\pi}{2}$.

Substituting the corresponding values in equation (1),

$$\int_0^\infty \frac{xdx}{1+x^6} = \int_0^{\pi/2} \frac{\tan\theta}{1+\tan^6\theta}(\sec^2\theta).d\theta$$

$$= \int_0^{\pi/2} \frac{\tan\theta\sec^2\theta\, d\theta}{(1+(\tan^3\theta)^2)}$$

$$= \int_0^{\pi/2} \frac{\tan\theta\sec^2\theta}{(\sec^3\theta)^2}d\theta \quad [\because 1+\tan^2\theta = \sec^2\theta]$$

$$= \int_0^{\pi/2} \frac{\tan\theta}{\sec^4\theta}d\theta = \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta}.\frac{1}{\sec^4\theta}d\theta$$

$$= \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta}.\cos^4\theta d\theta = \int_0^{\pi/2}\sin\theta\cos^3\theta d\theta$$

Since, $$\int_0^{\pi/2}\sin^m\theta\cos^n\theta\, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2},\frac{n+1}{2}\right) \quad \text{... (2)}$$

Substituting, $m = 1$ and $n = 3$ in equation (2),

$$\int_0^{\pi/2}\sin\theta\cos^3\theta\, d\theta = \frac{1}{2}\beta\left(\frac{1+1}{2},\frac{3+1}{2}\right)$$

$$= \frac{1}{2}\beta\left(\frac{2}{2},\frac{4}{2}\right) = \frac{1}{2}\beta(1, 2)$$

$$= \frac{1}{2}\frac{\Gamma(1)\Gamma(2)}{\Gamma(1+2)}$$

$$\left(\because \beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m)+(n)}\right)$$

$$= \frac{1}{2}\frac{\Gamma(1).\Gamma(2)}{\Gamma(3)}$$

$$= \frac{1}{2}\frac{\Gamma(1)\,\Gamma(2)}{(3-1)!} \quad [\because \Gamma(n) = (n-1)!]$$

$$= \frac{1}{2}\frac{\Gamma(1)\,\Gamma(2)}{2!}$$

$$= \frac{1}{2}.\frac{\Gamma(1)}{2} = \frac{1}{2}.\frac{1}{2} = \frac{1}{4}$$

$$\therefore \int_0^\infty \frac{x.dx}{(1+x^6)} = \frac{1}{4}$$

Q14. Evaluate $\int_0^\infty e^{-mx}(1-e^{-x})^n dx$, where m, n are positive constants.

Answer : May/June-17, Q8

Given integral is,

$$\int_0^\infty e^{-mx}(1-e^{-x})^n\,dx$$

$$\Rightarrow \int_0^\infty e^{-mx}(1-e^{-x})^n\,dx = \int_0^\infty (e^{-x})^m(1-e^{-x})^n\,dx$$

Let, $y = e^{-x} \Rightarrow dy = -e^{-x}dx$

Limits: $x = 0 \Rightarrow y = e^{-0} = 1$

$x = \infty \Rightarrow y = e^{-\infty} = 0$

$$\int_0^\infty e^{-mx}(1-e^{-x})^n\,dx = \int_1^0 \frac{y^m(1-y)^n}{-y}dy$$

$$= \int_0^1 y^{m-1}(1-y)^n dy$$

$$= \int_0^1 y^{m-1}(1-y)^{n+1-1} dy$$

$$= \beta(m, n+1)$$

Q15. Evaluate $\int_0^n x^n\left(1-\frac{x}{m}\right)^{m-1} dx$ in terms of beta function where m, n $\in$N.

Answer : Dec.-17, Q8

Note: In equation $\int_0^m x^n\left(1-\frac{x}{m}\right)^{m-1} dx$ is misprinted as $\int_0^n x^n\left(1-\frac{x}{m}\right)^{m-1} dx.$

Given integral is,

$$\int_0^m x^n\left(1-\frac{x}{m}\right)^{m-1} dx$$

Let $\left(1-\frac{x}{m}\right) = t \Rightarrow x = m(1-t)$

$$\frac{-1}{m}dx = dt$$

$$dx = -mdt$$

U.L : If $x = m$, $t = 0$

L.L : If $x = 0$, $t = 1$

Then,

$$\int_0^m x^n\left(1-\frac{x}{m}\right)^{m-1} dx = \int_1^0 (m(1-t))^n t^{m-1}(-mdt)$$

$$= -m\int_1^0 m^n(1-t)^n.t^{m-1}dt$$

$$= m\int_0^1 m^n(1-t)^{n+1-1}t^{m-1}dt$$

$$= m.m^n\int_0^1 t^{m-1}(1-t)^{n+1-1}dt$$

$$= m^{n+1}\beta(m, n+1)$$

$$\int_0^m x^n\left(1-\frac{x}{m}\right)^{m-1} dx = m^{n+1}\beta(m, n+1).$$

Q16. Define error function and complementary error function. List their properties.

Answer :

Error Function

The error function or the probability integral is defined by the relation,

$$erf(x) = \frac{2}{\pi}\int_0^x e^{-t^2}dt$$

Complementary Error Function

The complementary error function is defined as,

$$erfc(x) = 1 - erf(x)$$

(or)

$$erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$$

Properties

(i) $erf(-x) = -erf(x)$

(ii) $erf(0) = 0$

(iii) $erf(\infty) = 1$

(iv) $erf(-\infty) = -erf(\infty) = -1$

(v) $erf(x) + erf(-x) = 0$

(vi) $erfc(x) + erfc(-x) = 2$

Q17. Define error function. Prove that erf(–x) = – erf(x).

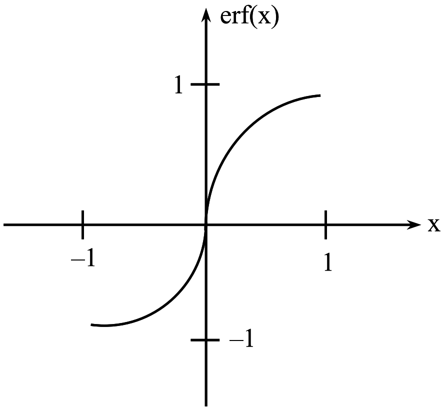
Answer : [Model Paper-3, Q8 | May/June-15, Q7]

Error Function

The error function is defined by the integral,

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt, -\infty < x < \infty$$

The graphical representation of error function is shown in below figure.



Figure

From the definition of error function,

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$\Rightarrow \quad erf(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-t^2} dt$$

Let,

$$t = -u \Rightarrow dt = -du$$

Lower Limit $t = 0 \Rightarrow u = 0$

Upper Limit $t = -x \Rightarrow u = x$

$$\therefore \quad erf(-x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} (-du)$$

$$= \frac{-2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$$

$$= \frac{-2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = -erf(x)$$

$$\therefore erf(-x) = -erf(x)$$

Q18. Define ordinary point of a differential equation.

Answer :

Ordinary Point

A differential equation of second order is given as,

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0$$

Where,

P_0, P_1, P_2 - Polynomials in x

A point $x = a$ is said to be an ordinary point of differential equation if and only if $P_0(a) \neq 0$.

Example

$$(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

Dividing above equation with $(1 + x^2)$,

$$\frac{d^2y}{dx^2} + \frac{x}{1+x^2}\frac{dy}{dx} - \frac{1}{1+x^2}y = 0$$

Here $x = 0$ is an ordinary point since $(1 + x^2) \neq 0$.

Q19. Define singular and regular singular points.

Answer : June-11, Q5

A differential equation of second order is given as,

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0$$

Where,

P_0, P_1, P_2 - Polynomials in x.

Singular Point

A point $x = a$ is said to be singular point of the differential equation if and only if $P_0(a) = 0$

Example

$$x^2\frac{d^2y}{dx^2} + (2x^2 - x)\frac{dy}{dx} + y = 0$$

Dividing above equation with 'x^2',

$$\frac{d^2y}{dx^2} + \frac{(2x^2 - x)}{x^2}\frac{dy}{dx} + \frac{y}{x^2} = 0$$

For $x = 0$, $x^2 = 0$

$\therefore$ $x = 0$ is a singular point since $x^2 = 0$.

Regular Singular Point

A singular point $x = a$ of a differential equation is said to be regular singular if it is in the form of,

$$\frac{d^2y}{dx^2} + \frac{Q_1(x)}{x-a}\frac{dy}{dx} + \frac{Q_2(x)}{(x-a)^2}y = 0$$

Where,

$Q_1(x)$, $Q_2(x)$ – Derivatives of all orders.

Alternatively, a singular point $x = a$ of differential equation $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(y) = 0$ is said to be regular if,

$(x - a)P(x)$, $(x - a)^2 Q(x)$ are analytic (i.e., not infinite).

Q20. Classify the singular points of a differential equation.

Answer :

The singular points of a differential equation are classified as,

(i) Regular singular points and

(ii) Irregular singular points.

(i) A point $x = a$, is said to be regular singular if for $x = a$, the values $(x - a)P$ and $(x - a)^2Q$ are not equal to infinity (∞).

(ii) A point $x = a$, is said to be irregular singular if for $x = a$, the values $(x - a)P$ and $(x - a)^2Q$ are equal to infinity (∞).

Q21. Find the singular points of x^2 y'' + xy'+ (x^2 – n^2) y = 0. Classify them.

Dec.-12, Q7

OR

Classify the singular points of, x^2y'' + xy' + (x^2 – n^2)y = 0, n is a constant.

Answer : Dec.-09/Jan.-10, Q7

Given differential equation is,

$$x^2y'' + xy' + (x^2 - n^2)\, y = 0 \qquad \text{... (1)}$$

Equation (1) can be written as,

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \qquad \text{... (2)}$$

Equation (2) is of the form,

$$\Rightarrow \quad P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)\, y = 0 \qquad \text{... (3)}$$

Comparing equation (3) with equation (2),

$$P_0(x) = x^2$$

For $P_0(x) = 0$

$$\Rightarrow \quad 0 = x^2 \quad \Rightarrow \quad x = 0$$

$\therefore$ $x = 0$ is a singular point.

From equation (1),

$$\frac{d^2y}{dx^2} + \frac{x}{x^2}\frac{dy}{dx} + \frac{x^2 - n^2}{x^2}y = 0$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \frac{x^2 - n^2}{x^2}y = 0 \qquad \text{... (4)}$$

Equation (4) is of the form,

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(y) = 0 \qquad \text{... (5)}$$

Comparing equation (5) with equation (4),

$$P(x) = \frac{1}{x}, \quad Q = \frac{1}{x^2}(x^2 - n^2)$$

For $x = 0$,

$$(x - a)\, P(x) = (x - 0)\ \frac{1}{x} = 1 \neq \infty$$

$$(x - a)^2\, Q(x) = (x - 0)^2\ \frac{1}{x^2}\ (x^2 - n^2) = x^2 - n^2 \neq \infty$$

$\therefore$ Equation (1) has regular singular point at $x = 0$.

Q22. Find the singular points of x^2y'' + (x + x^2)y' – y = 0 and classify them.

Answer : Jan.-12, Q5

Given differential equation is,

$$x^2y'' + (x + x^2)y' - y = 0 \qquad \text{... (1)}$$

Equation (1) can be written as,

$$x^2\frac{d^2y}{dx^2} + (x + x^2)\frac{dy}{dx} - y = 0 \qquad \text{... (2)}$$

Equation (2) is of the form,

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \qquad \text{... (3)}$$

Comparing equation (3) with equation (2),

$$P_0(x) = x^2$$

For $P_0(x) = 0$

$$x^2 = 0 \Rightarrow \quad x = 0$$

$\therefore$ $x = 0$ is a singular point.

From equation (1),

$$\frac{d^2y}{dx^2} + \frac{(x + x^2)}{x^2}\frac{dy}{dx} + \frac{1}{x^2}y = 0$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} + \left(\frac{1}{x} + 1\right)\frac{dy}{dx} + \frac{1}{x^2}y = 0 \qquad \text{... (4)}$$

Equation (4) is of the form,

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \qquad \text{... (5)}$$

Comparing equation (5) with equation (4),

$$P(x) = \left(\frac{1}{x} + 1\right), \qquad Q(x) = \frac{1}{x^2}$$

For $x = 0$,

$$(x - a)\, P(x) = (x - 0)\left(\frac{1}{x} + 1\right) = 1 + x \neq \infty$$

$$(x - a)^2\, Q(x) = (x - 0)^2\left(\frac{1}{x^2}\right) = 1 \neq \infty$$

$\therefore$ The given differential equation has regular singular point at $x = 0$

Q23. Determine the nature of the singular points of the differential equation x^2y'' + 9xy' + 6y = 0.

Answer : May/June-17, Q5

Given differential equation is,

$$x^2y'' + 9xy' + 6y = 0 \qquad \text{... (1)}$$

Equation (1) can be written as,

$$x^2\frac{d^2y}{dx^2} + 9x\frac{dy}{dx} + 6y = 0 \qquad \text{... (2)}$$

Equation (2) is of the form,

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \qquad \text{... (3)}$$

Comparing equations (2) and (3),

$P_0(x) = x^2$

For $P_0(x) = 0$

$x^2 = 0 \Rightarrow x = 0$

$\therefore$ $x = 0$ is a singular point.

From equation (2),

$$\frac{d^2y}{dx^2} + \frac{9}{x}\frac{dy}{dx} + \frac{6}{x^2}y = 0 \quad \text{... (4)}$$

Equation (4) is of the form,

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \quad \text{... (5)}$$

Comparing equations (4) and (5),

$$P(x) = \frac{9}{x}, Q(x) = \frac{6}{x^2}$$

For $a = 0$

$$(x-a)\,P(x) = (x-0)\,\frac{9}{x} = 9 \neq \infty$$

$$(x-a)^2\,Q(x) = (x-0)^2\,\frac{6}{x^2} = 6 \neq \infty$$

$\therefore$ Equation (1) has regular singular point at $x = 0$.

Q24. Determine the nature of the singular point of the differential equation x²y" – 5y' + 3x²y = 0.

Answer : Dec.-17, Q5

Given differential equation is,

$x^2y'' - 5y' + 3x^2y = 0$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} - \frac{5dy}{dx} + 3x^2y = 0 \quad \text{... (1)}$$

Equation (1) is of the form,

$$p_0(x)\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = 0 \quad \text{... (2)}$$

Comparing equations (1) and (2),

$p_0(x) = x^2$

For $p_0(x) = 0$, $x^2 = 0$

$\Rightarrow$ $x = 0$

$\therefore$ $x = 0$ is a singular point

From equation (1)

$$\frac{d^2y}{dx^2} - \frac{5}{x^2}\frac{dy}{dx} + 3y = 0 \quad \text{... (3)}$$

Equation (3) is of the form,

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + Q(x)y = 0 \quad \text{... (4)}$$

Comparing equations (3) and (4),

$$p(x) = \frac{-5}{x^2}, Q(x) = 3$$

For $x = 0$,

$$(x-a)\,p(x) = (x-0)\left(\frac{-5}{x^2}\right) = \frac{-5}{x} = \frac{-5}{0} = \infty$$

$$(x-a)^2\,Q(x) = (x-0)^2(3) = 3x^2 = 3(0)^2 \neq \infty.$$

$\therefore$ $x = 0$ is an irregular singular point.

Q25. Give the equation of power series expansion.

Answer :

The power series expansion is given as,

$$y(x) = C_0 + C_1(x-x_0) + C_2(x-x_0)^2 + C_3(x-x_0)^3 + ...$$

or

$$y(x) = \sum_{m=0}^{\infty} C_m(x-x_0)^2$$

Q26. Find the value of $P_n'(-1)$.

Answer : May/June-12, Q5

The Legendre's differential equation is given by,

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\,y = 0 \quad \text{... (1)}$$

Equation (1) can be written as,

$$(1-x^2)\,P_n''(x) - 2x\,P_n'(x) + n(n+1)\,P_n(x) = 0 \quad \text{... (2)}$$

Substituting $x = -1$ in equation (2),

$\Rightarrow$ $0 + 2\,P_n'(-1) + n(n+1)\,P_n(-1) = 0$

$\Rightarrow$ $2\,P_n'(-1) + n(n+1)\,(-1)^n\,P_n(1) = 0$

$[\because P_n(-1) = (-1)^n P_n(1)]$

$\Rightarrow$ $2\,P_n'(-1) + n(n+1)\,(-1)^n\,(1) = 0$

$[\because P_n(1) = 1]$

$\Rightarrow$ $2\,P_n'(-1) = -n(n+1)\,(-1)^n$

$\Rightarrow$ $2\,P_n'(-1) = (-1)^{n-1}\,n(n+1)$

$$\Rightarrow \quad P_n'(-1) = \frac{(-1)^{n-1}}{2}n(n+1)$$

$$\therefore P_n'(-1) = \frac{(-1)^{n-1}}{2}n(n+1)$$

Q27. Write the expression for Rodrigue's formula.

Answer :

The expression for Rodrigue's formula is given as,

$$P_n(x) = \frac{1}{2^n n!}\cdot\frac{d^n}{dx^n}(x^2-1)^n$$

Substituting different values for n,

(i) $P_0(x) = 1$

(ii) $P_1(x) = x$

(iii) $P_2(x) = \frac{1}{2}(3x^2-1)$

(iv) $P_3(x) = \frac{1}{2}(5x^3-3x)$

(v) $P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$

(vi) $P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$

Q28. Evaluate $4P_3(x) + 6P_2(x) + 3P_1(x)$ as a polynomial of x.

Answer : Dec.-17, Q6

Given expression is,

$$4P_3(x) + 6P_2(x) + 3P_1(x)$$

From Legendre's polynomials,

$$P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2-1),\ P_3(x) = \frac{5x^3-3x}{2}$$

$\Rightarrow$ $4P_3(x) + 6P_2(x) + 3P_1(x)$

$$= 4\left[\frac{5x^3-3x}{2}\right] + 6\left[\frac{3x^2-1}{2}\right] + 3(x)$$

$$= 2(5x^3-3x) + 3(3x^2-1) + 3x$$

$$= 10x^3 - 6x + 9x^2 - 3 + 3x$$

$$= 10x^3 + 9x^2 - 3x - 3$$

$\therefore$ $4P_3(x) + 6P_2(x) + 3P_1(x) = 10x^3 + 9x^2 - 3x - 3$.

Q29. Express $1 + x - x^2$ in terms of Legendre's polynomials $P_n(x)$.

Answer : Dec.-13, Q5

Given function is,

$$f(x) = 1 + x - x^2 \quad \text{... (1)}$$

From Rodrigue's formula,

$$P_0(x) = 1 \quad \text{... (2)}$$

$$P_1(x) = x \quad \text{... (3)}$$

And $P_2(x) = \frac{1}{2}(3x^2-1)$

$\Rightarrow$ $3x^2 - 1 = 2P_2(x)$

$\Rightarrow$ $3x^2 = 2P_2(x) + 1$

$\Rightarrow$ $x^2 = \frac{2P_2(x)+1}{3}$

$\Rightarrow$ $x^2 = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)$ $\quad [\because P_0(x) = 1]$ ... (4)

Substituting equations (2), (3) and (4) in equation (1),

$$f(x) = P_0(x) + P_1(x) - \left(\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right)$$

$\Rightarrow$ $f(x) = P_0(x) + P_1(x) - \frac{2}{3}P_2(x) - \frac{1}{3}P_0(x)$

$\Rightarrow$ $f(x) = P_0(x) + P_1(x) - \frac{2}{3}P_2(x) - \frac{1}{3}P_0(x)$

$\Rightarrow$ $f(x) = \left(1 - \frac{1}{3}\right)P_0(x) + P_1(x) - \frac{2}{3}P_2(x)$

$\Rightarrow$ $f(x) = \frac{2}{3}P_0(x) + P_1(x) - \frac{2}{3}P_2(x)$

$$\therefore\ f(x) = \frac{1}{3}\left[2P_0(x) - 3P_1(x) - 2P_2(x)\right]$$

Q30. Express $f(x) = 5x^3 + 6x^2 + 4$ in terms of Legendre polynomials.

Answer : May/June-17, Q6

Given function is,

$$f(x) = 5x^3 + 6x^2 + 4 \quad \text{... (1)}$$

From Legendre's polynomials,

$$P_0(x) = 1,\ P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2-1)$$

$\Rightarrow$ $x^2 = \frac{2}{3}P_2(x) - \frac{1}{3}P_0(x)$

$$P_3(x) = \frac{1}{2}(5x^3-3x)$$

$\Rightarrow$ $x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$

Substituting the corresponding values in equation (1),

$$f(x) = 5\left[\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\right] + 6\left[\frac{2}{3}P_2(x) - \frac{1}{3}P_0(x)\right] + 4P_0(x)$$

$$= 10P_3(x) + 3P_1(x) + 4P_2(x) + 2P_0(x) + 4P_0(x)$$

$$= 10P_3(x) + 4P_2(x) + 3P_1(x) + 6P_0(x)$$

$$\therefore f(x) = 10P_3(x) + 4P_2(x) + 3P_1(x) + 6P_0(x)$$

PART-B
ESSAY QUESTIONS WITH SOLUTIONS

4.1 GAMMA FUNCTIONS, BETA FUNCTIONS, RELATION BETWEEN BETA AND GAMMA FUNCTION

Q31. Define Gamma function. What is its reduction formula?

Answer :

Gamma Function

For answer refer Unit-4, Q.No. 1, Topic: Gamma Function.

Reduction or Recurrence Formula

$$\Gamma(n) = \int_0^\infty e^{-x}.x^{n-1}.dx$$

Substituting $n = n + 1$ in above equation,

$$\Gamma(n+1) = \int_0^\infty e^{-x}.x^{n+1-1}.dx$$

$$= \int_0^\infty e^{-x}.x^{n}.dx$$

$$= x^n \int_0^\infty e^{-x}.dx - \int_0^\infty \left((n.x^{n-1}) \int_0^\infty e^{-x}.dx \right) dx \quad \text{[Using integration by parts]}$$

$$= x^n.\frac{e^{-x}}{-1}\Bigg|_0^\infty - n\int_0^\infty x^{n-1}.\left(\frac{e^{-x}}{-1}\right)dx$$

$$= -x^n.e^{-x}\Big|_0^\infty + n\int_0^\infty x^{n-1}(e^{-x})dx$$

$$= 0 + n\int_0^\infty e^{-x}.x^{n-1}.dx \qquad \left(\because \lim_{x\to\infty} x^n.e^{-x} = \lim_{x\to\infty}\frac{x^n}{e^x} = 0 \quad \lim_{x\to 0}\frac{x^n}{e^x} = \frac{0}{1} = 0 \right)$$

$$= n\int_0^\infty e^{-x}.x^{n-1}dx$$

$$= n\Gamma(n)$$

$$\therefore \quad \Gamma(n+1) = n\Gamma(n)$$

This is known as recurrence or reduction formula of Gamma function.

Q32. Express $\int_0^1 x^m(1-x^n)^p\,dx$ in terms of Gamma function and evaluate $\int_0^1 x^5(1-x^3)^8\,dx$.

Answer :

Given integral is,

$$\int_0^1 x^m(1-x^n)^p.dx$$

Let, $x^n = t \Rightarrow x = t^{1/n}$

$$dx = \frac{1}{n}t^{\frac{1}{n}-1}dt$$

$$\therefore \int_0^1 x^m(1-x^n)^p.dx = \int_0^1 (t^{1/2})^m(1-t)^p.\frac{1}{n}t^{\frac{1}{n}-1}.dt$$

$$= \frac{1}{n}\int_0^1 t^{\frac{m+1}{n}-1}(1-t)^p.dt$$

$$= \frac{1}{n}\int_0^1 t^{\frac{m+1}{n}-1}(1-t)^{(p+1)-1}.dt$$

$$= \frac{1}{n}\left[\beta\left(\frac{m+1}{n}, p+1\right)\right]$$

$$\left(\because \int_0^1 x^{m-1}(1-x)^{n-1}dx = \beta(m,n)\right)$$

$$= \frac{1}{n}.\frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)}$$

$$\left(\because \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right)$$

$$\therefore \int_0^1 x^m(1-x^n)^p dx = \frac{1}{n}.\frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)} \quad ... (1)$$

$$\int_0^1 x^5(1-x^3)^8 dx$$

Here, $m = 5$, $n = 3$, $p = 8$

Substituting the corresponding values in equation (1),

$$\int_0^1 x^5(1-x^3)^8 dx = \frac{1}{3}.\frac{\Gamma\left(\frac{5+1}{3}\right)\Gamma(8+1)}{\Gamma\left(\frac{5+1}{3}+8+1\right)}$$

$$= \frac{1}{3}.\frac{\Gamma\left(\frac{6}{3}\right)\Gamma(9)}{\Gamma(11)}$$

$$= \frac{1}{3}.\frac{\Gamma(2)\Gamma(9)}{\Gamma(11)}$$

$$= \frac{1}{3}.\frac{1!\,8!}{10!} \quad [\because \Gamma(n) = (n-1)!]$$

$$= \frac{1}{3}.\frac{1\times 8!}{10\times 9\times 8!}$$

$$= \frac{1}{270}$$

$$\therefore \int_0^1 x^5(1-x^3)^8 dx = \frac{1}{270}$$

Q33. Show that $2n\Gamma\left(n+\frac{1}{2}\right)$ = 1.3.5...(2n – 1) $\sqrt{\pi}$

Where n is a positive integer.

Answer : **Model Paper-1, Q14(a)**

Given that,

$$2^n.\Gamma\left(n+\frac{1}{2}\right) = 1.3.5......(2n-1)\sqrt{\pi} \quad ... (1)$$

Gamma function in terms of 'n' is given by,

$$\Gamma(n) = (n-1)\Gamma(n-1) \quad ... (2)$$

Substituting $n = n + \frac{1}{2}$ in equation (2),

$$\Gamma\left(n+\frac{1}{2}\right) = \left(n+\frac{1}{2}-1\right)\Gamma\left(n+\frac{1}{2}-1\right)$$

$$= \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{1}{2}-1\right)\Gamma\left(n-\frac{1}{2}-1\right)$$

[$\because$ From equation (2)]

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right)$$

[$\because$ From equation (2)]

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\Gamma\left(n-\frac{5}{2}\right)....$$

[$\because$ From equation (2)]

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)...\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right)$$

$$= \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\left(\frac{2n-5}{2}\right)..\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}$$

$$\left(\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$$

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)(2n-3)(2n-5)...5.3.1}{2^n}.\sqrt{\pi}$$

$$\Rightarrow 2^n\Gamma\left(n+\frac{1}{2}\right) = (2n-1)\,(2n-3)\,(2n-5)\,...\,5.3.1\,\sqrt{\pi}$$

$$\therefore 2^n\Gamma\left(n+\frac{1}{2}\right) = 1.3.5...(2n-1)\sqrt{\pi}$$

Q34. Evaluate $\int_e^{\infty} \sqrt{x}\,e^{-x^3}$

Answer :

Given integral is,

$$\int_0^{\infty} \sqrt{x}.e^{-x^3} dx$$

Let,

$x^3 = a$

$\Rightarrow \quad x = a^{1/3}$

Differentiating above equation with respect to a,

$$dx = \frac{1}{3}a^{\frac{1}{3}-1}da$$

$$dx = \frac{1}{3}a^{\frac{-2}{3}}da$$

Limits

For $x = 0, \; a = 0$

For $x = \infty, \; a = \infty$

$\therefore$ Limits of a are from 0 to ∞

$$\therefore \int_0^\infty \sqrt{x}.e^{-x^3}.dx = \int_0^\infty (a^{1/3})^{1/2}e^{-a}\left(\frac{1}{3}a^{-2/3}da\right)$$

$$= \frac{1}{3}\int_0^\infty a^{\frac{1}{6}}e^{-a}a^{\frac{-2}{3}}da$$

$$= \frac{1}{3}\int_0^\infty a^{\frac{1}{6}-\frac{2}{3}}e^{-a}da$$

$$= \frac{1}{3}\int_0^\infty a^{\frac{-1}{2}}e^{-a}da$$

$$= \frac{1}{3}\int_0^\infty a^{\frac{1}{2}-1}e^{-a}da$$

$$= \frac{1}{3}\Gamma\left(\frac{1}{2}\right) \qquad \left(\because \int_0^\infty x^{n-1}e^{-x}dx = \Gamma(n)\right)$$

$$= \frac{1}{3}\times\sqrt{\pi} = \frac{\sqrt{\pi}}{3} \qquad \left(\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$$

$$\therefore \int_0^\infty \sqrt{x}e^{-x^3}dx = \frac{\sqrt{\pi}}{3}$$

Q35. Prove that β(m + 1, n) + β(m, n + 1) = β(m, n).

OR

Prove that β(m, n) = β(m + 1, n) + β(m, n + 1).

OR

Show that β(m, n + 1) + β(m + 1, n) = β(m, n).

Answer : **(Jan.-12, Q16(a) | June-10, Q16(b))**

Given that,

$\beta(m, n) = \beta(m + 1, n) + \beta(m, n + 1)$

Consider,

$\beta(m + 1, n) + \beta(m, n + 1)$

From the definition of beta function,

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

$\beta(m + 1, n) + \beta(m, n + 1)$

$$= \int_0^1 x^{m+1-1}(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^{n+1-1}dx$$

$$= \int_0^1 x^m(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^n dx$$

$$= \int_0^1 [x^m(1-x)^{n-1} + x^{m-1}(1-x)^n]dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}[x + (1-x)]dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}dx = \beta(m, n)$$

$\therefore \beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$

Q36. Show that $\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$.

Answer :

Given that,

$$\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$$

The general expression for β-function is given by,

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}.dx$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}.dx \qquad \text{... (1)}$$

$[\because \; \beta(m, n) = \beta(n, m)]$

Let, $x = \dfrac{1}{1+y}$ ... (2)

$$\Rightarrow \quad 1 - x = 1 - \frac{1}{1+y}$$

$$= \frac{1+y-1}{1+y}$$

$$\therefore \quad 1 - x = \frac{y}{1+y} \qquad \text{... (3)}$$

Differentiating equation (2), with respect to x,

$$dx = -\frac{1}{(1+y)^2}.(1).dy$$

$$\Rightarrow \quad dx = \frac{-1}{(1+y)^2}.dy \qquad \text{... (4)}$$

Limits

For $x = 0$, equation (2) becomes,

$$0 = \frac{1}{1+y}$$

$$\Rightarrow \quad \frac{1}{0} = 1 + y \quad \Rightarrow \quad \infty = 1 + y$$

$\therefore y = \infty$

For $x = 1$, equation (2) becomes,

$$1 = \frac{1}{1+y}$$

$$\Rightarrow \quad \frac{1}{1} = 1 + y \quad \Rightarrow \quad 1 = 1 + y$$

$$\therefore y = 0$$

∴ Limits are from ∞ to 0.

Substituting the corresponding values in equation (1),

$$\beta(m, n) = \int_{\infty}^{0} \left(\frac{1}{1+y}\right)^{n-1} \left(\frac{y}{1+y}\right)^{m-1} \left(-\frac{1}{(1+y)^2}.dy\right)$$

$$= \int_{\infty}^{0} \frac{(1)^{n-1}}{(1+y)^{n-1}} \frac{(y)^{m-1}}{(1+y)^{m-1}} \left(\frac{-dy}{(1+y)^2}\right)$$

$$= \int_{\infty}^{0} \frac{1 \times y^{m-1}(-dy)}{(1+y)^{n-1+m-1+2}}$$

$$= \int_{\infty}^{0} \frac{-y^{m-1}.dy}{(1+y)^{m+n-2+2}}$$

$$\beta(m, n) = \int_{0}^{\infty} \frac{y^{m-1}.dy}{(1+y)^{m+n}} \quad \text{... (5)}$$

Equation (5) can be expressed in terms of 'x' as,

$$\beta(m,n) = \int_{0}^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}.dx$$

$$\therefore \quad \beta(m, n) = \int_{0}^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

Q37. Show that $\beta\left(m, \frac{1}{2}\right) = 2^{2m-1}\beta(m, n)$

Answer : April-16, Q14(a)

Given that,

$$\beta\left(m, \frac{1}{2}\right) = 2^{2m-1}\beta(m, n)$$

The general expression for 'β' in terms of 'θ' is given by,

$$\beta(m, n) = 2\int_{0}^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \quad \text{... (1)}$$

Substituting $n = \frac{1}{2}$ in equation (1),

$$\beta\left(m, \frac{1}{2}\right) = 2\int_{0}^{\pi/2} \sin^{2m-1}\theta \, . \, \cos^{\left[2\left(\frac{1}{2}\right)-1\right]} d\theta$$

$$= 2\int_{0}^{\pi/2} \sin^{2m-1}\theta.d\theta$$

$$\therefore \quad \beta\left(m, \frac{1}{2}\right) = 2\int_{0}^{\pi/2} \sin^{2m-1}.\theta.d\theta \quad \text{... (2)}$$

Substituting $n = m$ in equation (1),

$$\beta(m, m) = 2\int_{0}^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta \, d\theta$$

$$= 2\int_{0}^{\pi/2} (\sin\theta \cos\theta)^{2m-1} d\theta$$

$$= \int_{0}^{\pi/2} 2(\sin\theta \cos\theta)^{2m-1} d\theta \quad \text{... (3)}$$

$$= \frac{1}{2^{2m-2}} \int_{0}^{\pi/2} (2^{2m-2})\,(2)\,(\sin\theta \cos\theta)^{2m-1} d\theta$$

$$= \frac{1}{2^{2m-2}} \int_{0}^{\pi/2} 2^{2m-2+1}(\sin\theta\cos\theta)^{2m-1}.d\theta$$

$$= \frac{1}{2^{2m-2}} \int_{0}^{\pi/2} 2^{2m-1} (\sin\theta \cos\theta)^{2m-1} d\theta$$

$$= \frac{1}{2^{2m-2}} \int_{0}^{\pi/2} (2\sin\theta \cos\theta)^{2m-1} d\theta$$

$$\Rightarrow \beta(m, m) = \frac{1}{2^{2m-2}} \int_{0}^{\pi/2} (\sin 2\theta)^{2m-1} d\theta \quad \text{... (4)}$$

Let, $2\theta = \phi$

Differentiating above equation with respect to 'θ',

$$2.d\theta = d\phi$$

$$\Rightarrow d\theta = \frac{d\phi}{2}$$

Limits: For $\theta = 0,\ 2(0) = \phi \Rightarrow \phi = 0$

For $\theta = \frac{\pi}{2}, 2\left(\frac{\pi}{2}\right) = \phi \Rightarrow \phi = \pi$

∴ Limits are from 0 to π.

Substituting the corresponding values in equation (4),

$$\beta(m, m) = \frac{1}{2^{2m-2}} \int_{0}^{\pi} (\sin\phi)^{2m-1} \left(\frac{d\phi}{2}\right)$$

$$= \frac{1}{2.2^{2m-2}} \int_{0}^{\pi} \sin^{2m-1}\phi \, d\phi$$

$$= \frac{1}{2^{2m-2+1}} \int_{0}^{\pi} \sin^{2m-1}\phi \, d\phi$$

$$= \frac{1}{2^{2m-1}} \int_{0}^{\pi} \sin^{2m-1}\phi \, d\phi$$

$$\beta(m, m) = \frac{1}{2^{2m-1}} \, 2\int_{0}^{\pi/2} \sin^{2m-1}\phi \, d\phi$$

$$\left[\because \int_{0}^{\pi} \sin\theta \, d\theta = 2\int_{0}^{\pi/2} \sin\theta \, d\theta\right]$$

$$\Rightarrow 2^{2m-1}\,\beta(m, m) = 2\int_0^{\pi/2} \sin^{2m-1}\phi\, d\phi$$

Replacing ϕ with θ,

$$2^{2m-1}\,\beta(m, m) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\, d\theta$$

$$\Rightarrow \quad 2^{2m-1}\,\beta(m, m) = \beta\left(m, \frac{1}{2}\right) \quad [\because \text{From equation (2)}]$$

$$\therefore\ \beta\left(m, \frac{1}{2}\right) = 2^{2m-1}\beta(m, n)$$

Q38. Prove that $\beta(m, n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

OR

Show that $\beta(m, n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

OR

Derive a relation between Beta and Gamma function.

Answer : **[Model Paper-3, Q14(a) | June-14, Q14(a)]**

The general expression for 'Γ' function is given by,

$$\Gamma(m) = \int_0^\infty e^{-x}.x^{m-1}.dx \quad \text{... (1)}$$

Let, $x = yt$

Differentiating above equation with respect to 'x',

$$dx = y.dt$$

Limits

For $x = 0$, $yt = 0 \Rightarrow y = 0$

For $x = \infty$, $yt = \infty \Rightarrow y = \infty$

$\therefore$ Limits are from 0 to ∞.

Substituting the corresponding values in equation (1),

$$\Gamma(m) = \int_0^\infty e^{-yt}.y^{m-1}.t^{m-1}.y.dt$$

$$\Gamma(m) = \int_0^\infty e^{-yt}.y^{m}.t^{m-1}.dt \quad \text{... (2)}$$

Equation (2) can also be expressed in terms of 'x'as,

$$\Gamma(m) = \int_0^\infty e^{-yx}.y^{m}.x^{m-1}.dx$$

$$\Rightarrow \quad \frac{\Gamma(m)}{y^m} = \int_0^\infty e^{-yx}.x^{m-1}.dx \quad \text{... (3)}$$

Multiplying on both sides by $\int_0^\infty e^{-y}.y^{m+n-1}.dy$,

$$\frac{\Gamma(m)}{y^m}\int_0^\infty e^{-y}.y^{m+n-1}dy = \int_0^\infty e^{-yx}.x^{m-1}dx\int_0^\infty (e^{-y}.y^{m+n-1})dy$$

$$\Rightarrow \quad \Gamma(m)\int_0^\infty e^{-y}.y^{n-1}dy = \int_0^\infty\int_0^\infty e^{-y(1+x)}.y^{m+n-1}.x^{m-1}.dx.dy$$

$$\Gamma(m)\Gamma(n) = \int_0^\infty\left\{\int_0^\infty e^{-y(1+x)}.y^{m+n-1}dy\right\}x^{m-1}.dx$$

$$= \int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}}.x^{m-1}.dx$$

[$\because$ From equation (3)]

$$= \Gamma(m+n)\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$$

$$= \Gamma(m+n).\beta(m, n)$$

$$\left(\because \beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx\right)$$

$$\Rightarrow \quad \Gamma(m).\Gamma(n) = \Gamma(m+n).\beta(m, n)$$

$$\therefore \quad \beta(m, n) = \frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}$$

Q39. Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Answer : **April-16, Q14(b)**

Given function is,

$$\Gamma\left(\frac{1}{2}\right)$$

The relation between 'β' and 'Γ' function is,

$$\beta(m, n) = \frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)} \quad \text{... (1)}$$

Substituting $m = n = \frac{1}{2}$ in equation (1),

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{\left(\Gamma\left(\frac{1}{2}\right)\right)^2}{\Gamma(1)} = \left(\Gamma\left(\frac{1}{2}\right)\right)^2 \quad \text{... (2)}$$

The general expression for β-function is,

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \quad \text{... (3)}$$

Substituting $m = \frac{1}{2}$ and $n = \frac{1}{2}$ in equation (3),

$$\beta\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{\frac{1}{2}-1}dx$$

$$\Rightarrow \quad \beta\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^1 x^{\frac{-1}{2}}(1-x)^{\frac{-1}{2}}dx \quad \text{... (4)}$$

Let, $x = \sin^2\theta \Rightarrow \sin\theta = \sqrt{x} \Rightarrow \theta = \sin^{-1}\sqrt{x}$

$\Rightarrow \quad dx = 2\sin\theta\cos\theta$

Limits For $x = 0$, $\theta = \sin^{-1}(\sqrt{0})$

$\Rightarrow \quad \theta = \sin^{-1}0 \Rightarrow \theta = \sin^{-1}(\sin 0) \Rightarrow \theta = 0$

For $x = 1$, $\theta = \sin^{-1}(\sqrt{1}) \Rightarrow \theta = \sin^{-1}(1)$

$\Rightarrow \quad \theta = \sin^{-1}\left(\sin\frac{\pi}{2}\right) \Rightarrow \theta = \frac{\pi}{2}$

$\therefore$ Limits are from 0 to $\frac{\pi}{2}$.

Substituting the corresponding values in (4),

$$\beta\left(\frac{1}{2}, \frac{1}{2}\right) = \int_0^{\pi/2} (\sin^2\theta)^{-1/2}(1-\sin^2\theta)^{-1/2}\, 2\sin\theta\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} (\sin\theta)^{-1}.(\cos^2\theta)^{-1/2}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \frac{1}{\sin\theta}.\frac{1}{\cos\theta}.\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} d\theta = 2[\theta]_0^{\pi/2}$$

$$= 2\left[\frac{\pi}{2} - 0\right] = 2.\frac{\pi}{2} = \pi$$

$$\therefore \beta\left(\frac{1}{2}, \frac{1}{2}\right) = \pi \quad \text{... (5)}$$

Comparing equations (2) and (5),

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\therefore \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Q40. Prove that $\beta(n, n) = \dfrac{\sqrt{\pi}\,\Gamma(n)}{2^{2n-1}\Gamma\left(n+\frac{1}{2}\right)}$.

Answer :

Given that,

$$\beta(n, n) = \frac{\sqrt{\pi}\Gamma(n)}{2^{2n-1}.\Gamma\left(n+\frac{1}{2}\right)}$$

The general expression for β-function is given by,

$$\beta(m, n) = 2\int_0^{\frac{\pi}{2}} (\sin x)^{2m-1}(\cos x)^{2n-1}dx \quad \text{... (1)}$$

Substituting $m = n$ in equation (1),

$$\beta(n, n) = 2\int_0^{\frac{\pi}{2}} (\sin^{2n-1}x)(\cos^{2n-1}x)dx$$

$$= 2\int_0^{\frac{\pi}{2}} [\sin x\cos x]^{2n-1}dx$$

$$= 2\int_0^{\frac{\pi}{2}} \left[\frac{(2\sin x\cos x)}{2}\right]^{2n-1}dx$$

$$= \frac{2}{2^{2n-1}}\int_0^{\frac{\pi}{2}} (\sin 2x)^{2n-1}dx \quad \text{... (2)}$$

Let, $2x = \theta$

Differentiating on both sides with respect to 'x',

$$dx = \frac{d\theta}{2}$$

Limits

For $x = 0$, $\theta = 0$

For $x = \frac{\pi}{2}$, $\theta = \pi$

$\therefore$ Limits are from 0 to π.

Substituting the corresponding values in equation (2),

$$\beta(n, n) = \frac{2}{2^{2n-1}}\int_0^{\pi} (\sin\theta)^{2n-1}\frac{d\theta}{2}$$

$$\Rightarrow \quad \beta(n, n) = \frac{2}{2^{2n-1}}\times\frac{1}{2}\int_0^{2\left(\frac{\pi}{2}\right)} (\sin\theta)^{2n-1}d\theta$$

$$\Rightarrow \quad \beta(n, n) = \frac{2}{2^{2n-1}}\times\frac{2}{2}\int_0^{\frac{\pi}{2}} (\sin\theta)^{2n-1}(\cos\theta)^0\, d\theta$$

$$\left[\because \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx\right]$$

$$\Rightarrow \quad \beta(n, n) = \frac{2}{2^{2n-1}}.\frac{\Gamma\left(\frac{2n-1+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{2n-1+0+2}{2}\right)}$$

$$\because \int_0^{\frac{\pi}{2}} \sin^p\cos^q x\,dx = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

$$\Rightarrow \quad \beta(n, n) = \frac{1}{2^{2n-1}}.\frac{\Gamma(n)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)}$$

$$\Rightarrow \quad \beta(n, n) = \frac{\Gamma(n)\sqrt{\pi}}{2^{2n-1}\Gamma\left(n+\frac{1}{2}\right)} \quad \left(\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$$

$$\therefore \quad \beta(n, n) = \frac{\Gamma(n)\sqrt{\pi}}{2^{2n-1}\Gamma\left(n+\frac{1}{2}\right)}$$

Q41. Prove that $\beta(m, m)\,\beta\left(m+\frac{1}{2}, m+\frac{1}{2}\right) = \frac{\pi}{m2^{4m-1}}$.

Answer : Model Paper-2, Q14(a)

Given that,

$$\beta(m,m).\beta\left(m+\frac{1}{2}, m+\frac{1}{2}\right) = \frac{\pi}{m.2^{4m-1}} \quad \text{... (1)}$$

The relation between 'β' and 'Γ' function is,

$$\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad \text{... (2)}$$

Substituting $m = n = m + \frac{1}{2}$ in equation (2),

$$\beta\left(m+\frac{1}{2}, m+\frac{1}{2}\right) = \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}+m+\frac{1}{2}\right)}$$

$$= \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(2m+1)}$$

$$= \frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}{2m\,\Gamma(2m)}$$

$[\because \Gamma(m+1) = m\,\Gamma(m)]$

$$= \frac{\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m)2m\,\Gamma(2m)}$$

$$= \frac{1}{2m\Gamma(m)}.\frac{\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(2m)}.\Gamma\left(m+\frac{1}{2}\right)$$

$$= \frac{1}{2m\Gamma(m)}.\frac{\sqrt{\pi}\,\Gamma(2m)\Gamma\left(m+\frac{1}{2}\right)}{2^{2m-1}\Gamma(2m)}$$

$$\left[\because \Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)\right]$$

$$= \frac{1}{2m\Gamma(m)} \times \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma\left(m+\frac{1}{2}\right)$$

$$= \frac{\sqrt{\pi}}{m.2^{2m}\Gamma(m)}\Gamma\left(m+\frac{1}{2}\right)$$

Multiplying and dividing with $\Gamma(m)$,

$$= \frac{\sqrt{\pi}}{m.2^{2m}\Gamma(m)}.\frac{\Gamma(m).\Gamma\left(m+\frac{1}{2}\right)}{\Gamma(m)}$$

$$= \frac{\sqrt{\pi}}{m2^{2m}\Gamma(m)}.\frac{1}{\Gamma(m)}.\frac{\sqrt{\pi}}{2^{2m-1}}.\Gamma(2m)$$

$$\left(\because \Gamma m\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)\right)$$

$$= \frac{\pi}{m\,2^{4m-1}}.\frac{\Gamma(2m)}{\Gamma(m)\Gamma(m)}$$

$$= \frac{\pi}{m\,2^{4m-1}}.\frac{1}{\beta(m,m)} \qquad \left[\because \beta(m,m) = \frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)}\right]$$

$$\Rightarrow \beta\left(m+\frac{1}{2}, m+\frac{1}{2}\right) = \frac{\pi}{m2^{4m-1}}.\frac{1}{\beta(m, m)}$$

$$\therefore\ \beta(m, m)\,\beta\left(m+\frac{1}{2}, m+\frac{1}{2}\right) = \frac{\pi}{m2^{4m-1}}$$

Q42. Show that $\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta.\cos^{2n-1}\theta\, d\theta$ and deduce that, $\int_0^{\pi/2} \sin^n\theta\, d\theta = \int_0^{\pi/2} \cos^n\theta\, d\theta = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{n+2}{2}\right)}$.

Answer :

Given that,

$$\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}.\theta.\cos^{2n-1}\theta.d\theta$$

The general expression for β-function is given by,

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx = \beta(m, n) \quad \text{... (1)}$$

Let, $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta.d\theta$

Limits

For $x = 0$, $0 = \sin^2\theta \Rightarrow \sin\theta = 0 \Rightarrow \sin\theta = \sin 0 \Rightarrow \theta = 0$

For $x = 1$, $1 = \sin^2\theta \Rightarrow \sin\theta = 1 \Rightarrow \sin\theta = \sin\frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{2}$

$\therefore$ Limits are from 0 to $\frac{\pi}{2}$

Substituting the corresponding values in equation (1),

$$\int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1}2\sin\theta\cos\theta\, d\theta = \beta(m, n)$$

$$\Rightarrow \quad 2\int_0^{\pi/2} \sin^{2m-2}\theta(\cos^2\theta)^{n-1}\sin\theta\cos\theta\, d\theta = \beta(m, n)$$

$[\because 1 - \sin^2\theta = \cos^2\theta]$

$$\Rightarrow \quad 2\int_0^{\pi/2} \sin^{2m-2}\theta\sin\theta(\cos^{2n-2}\theta)\cos\theta\, d\theta = \beta(m, n)$$

$$\Rightarrow \quad 2\int_0^{\pi/2}\left(\frac{\sin^{2m-1}\theta}{\sin\theta}\right)\sin\theta\left(\frac{\cos^{2n-1}\theta}{\cos\theta}\right)\cos\theta\, d\theta = \beta(m, n)$$

$$\Rightarrow \quad 2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta.d\theta = \beta(m, n)$$

$$\therefore \quad \beta(m, n) = 2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta.d\theta \qquad \text{... (2)}$$

Deduction

Let, $2m - 1 = n$ and $2n - 1 = 0$

$$\Rightarrow \quad 2m = n + 1 \text{ and } 2n = 1$$

$$\Rightarrow \quad m = \frac{n+1}{2} \text{ and } n = \frac{1}{2}$$

$$\therefore \quad \int_0^{\pi/2}\sin^n\theta\cos^0\theta.d\theta = \frac{1}{2}\beta(m,n)$$

$$\Rightarrow \quad \int_0^{\pi/2}\sin^n\theta.1.d\theta = \frac{1}{2}\beta(m,n)$$

$$\Rightarrow \quad \int_0^{\pi/2}\sin^n\theta\, d\theta = \frac{1}{2}.\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \qquad \left[\because \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right]$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}+\frac{1}{2}\right)}$$

$$\therefore \quad \int_0^{\pi/2}\sin^n\theta\, d\theta = \frac{1}{2}\frac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \quad \left(\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right) \qquad \text{... (3)}$$

Similarly,

Let, $2m - 1 = 0$ and $2n - 1 = n$

$$\Rightarrow \quad m = \frac{1}{2} \text{ and } n = \frac{n+1}{2}$$

$$\therefore \quad \frac{1}{2}\beta(m, n) = \int_0^{\pi/2}\sin^0\theta\cos^n\theta\, d\theta = \int_0^{\pi/2}\cos^n\theta\, d\theta$$

$$\Rightarrow \quad \frac{1}{2}\beta(m, n) = \int_0^{\pi/2}\cos^n\theta\, d\theta$$

$$\Rightarrow \quad \frac{1}{2}\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \int_0^{\pi/2}\cos^n\theta\, d\theta$$

$$\frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{n+1}{2}\right)} = \int_0^{\pi/2}\cos^n\theta\, d\theta$$

$$\Rightarrow \quad \frac{1}{2}\frac{\sqrt{\pi}\;\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} = \int_0^{\pi/2}\cos^n\theta\, d\theta$$

From equations (3) and (4),

$$\int_0^{\pi/2} \sin^n \theta\, d\theta = \int_0^{\pi/2} \cos^n \theta\, d\theta = \frac{1}{2}\frac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$$

$$\therefore \quad \int_0^{\pi/2} \sin^n \theta\, d\theta = \int_0^{\pi/2} \cos^n \theta\, d\theta = \frac{1}{2}\frac{\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$$

Q43. Show that $\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

OR

Show that $\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$

Answer :

Given that,

$$\int_0^{\pi/2} \sin^p \theta.\cos^q \theta.d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Consider,

$$\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta\, d\theta = \int_0^{\frac{\pi}{2}} [\sin\theta]^{p-1}[\cos\theta]^{q-1}(\sin\theta\cos\theta\, d\theta)$$

$$= \int_0^{\frac{\pi}{2}} [\sin^2\theta]^{\frac{p-1}{2}}[\cos^2\theta]^{\frac{q-1}{2}}(\sin\theta\cos\theta\, d\theta)$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} [\sin^2\theta]^{\frac{p-1}{2}}[1-\sin^2\theta]^{\frac{q-1}{2}}(2\sin\theta\cos\theta\, d\theta) \quad \text{... (1)}$$

Let, $\sin^2\theta = v$

Differentiating with respect to 'θ',

$2 \sin\theta \cos\theta\, d\theta = dv$

Limits

For $\theta = 0 \Rightarrow \sin^2 0 = v \Rightarrow v = 0$

For $\theta = \frac{\pi}{2} \Rightarrow \sin^2\frac{\pi}{2} = v \Rightarrow (1)^2 = v \Rightarrow v = 1$

∴ Limits are from 0 to 1.

Substituting the corresponding values in equation (1),

$$\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta\, d\theta = \frac{1}{2}\int_0^1 v^{\frac{p-1}{2}}(1-v)^{\frac{q-1}{2}}\, dv$$

$$= \frac{1}{2}\int_0^1 v^{\frac{p+1}{2}-1}(1-v)^{\frac{q+1}{2}-1}\, dv = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$\left(\because \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \quad \text{Where, } m = \frac{p+1}{2}, n = \frac{q+1}{2}\right)$$

$$\therefore \quad \int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

The relation between beta and gamma function is,

$$\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad \text{... (2)}$$

Substituting the corresponding values in equation (2),

$$\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+1}{2}+\frac{q+1}{2}\right)}$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

Q44. Express the integral $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$ in terms of Gamma functions.

Answer : June-11, Q16(a)

Given integral is,

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}}$$

Let, $x^4 = v \Rightarrow x = v^{\frac{1}{4}}$

Differentiating on both sides with respect to x,

$$dx = \frac{1}{4} v^{\frac{1}{4}-1} dv$$

$$\Rightarrow \quad dx = \frac{1}{4} v^{\frac{-3}{4}} dv$$

Limits

For $x = 0$, $v = (0)^4 = 0$

For $x = 1$, $v = (1)^4 = 1$

$\therefore$ Limits are from 0 to 1.

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_0^1 \frac{1}{\sqrt{1-v}}\left(\frac{1}{4} v^{\frac{-3}{4}} dv\right)$$

$$= \frac{1}{4}\int_0^1 (1-v)^{-1/2} v^{-3/4} dv$$

$$= \frac{1}{4}\int_0^1 v^{\frac{1}{4}-1}(1-v)^{\frac{1}{2}-1} dv$$

$$= \frac{1}{4}\beta\left(\frac{1}{4},\frac{1}{2}\right) \qquad \left(\because \int_0^1 x^{m-1}(1-x)^{n-1} dx = \beta(m,n)\right)$$

$$= \frac{1}{4}\cdot\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)} \qquad \left(\because \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right)$$

$$= \frac{1}{4}\cdot\frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{1+2}{4}\right)} \qquad \left(\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$$

$$= \frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{4\Gamma\left(\frac{3}{4}\right)}$$

$$\therefore \int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{4\Gamma\left(\frac{3}{4}\right)}$$

Q45. Evaluate $\int_0^1 \frac{dx}{(1-x^n)^{1/n}}$

Answer : May/June-12, Q16(a)

Given integral is,

$$\int_0^1 \frac{dx}{(1-x^n)^{1/n}}$$

Let, $x^n = \sin^2\theta \Rightarrow x = \sin^{2/n}\theta$

$$dx = \frac{2}{n}\sin^{\frac{2}{n}-1}\theta.\cos\theta \, d\theta$$

Limits

For $x = 0 \Rightarrow \theta = 0$

For $x = 1 \Rightarrow \theta = \frac{\pi}{2}$

$\therefore$ Limits are from 0 to $\frac{\pi}{2}$

$$\therefore \int_0^1 \frac{dx}{(1-x^n)^{1/n}} = \int_0^{\frac{\pi}{2}} \frac{1}{(1-\sin^2\theta)^{1/n}}.\frac{2}{n}\sin^{\frac{2}{n}-1}\theta.\cos\theta \, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{(\cos^2\theta)^{1/n}}.\frac{2}{n}\sin^{\frac{2}{n}-1}\theta.\cos\theta \, d\theta$$

$$\int_0^1 \frac{dx}{(1-x^n)^{1/n}} = \frac{2}{n}\int_0^{\pi/2} \sin^{\frac{2}{n}-1}\theta.\cos^{1-\frac{2}{n}}\theta \, d\theta$$

From the definition of gamma function,

$$\int_0^{\pi/2} \sin^p \theta \cos^q \theta \, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

Here, $p = \frac{2}{n} - 1$, $q = 1 - \frac{2}{n}$

$$\therefore \int_0^1 \frac{dx}{(1-x^n)^{1/n}} = \frac{2}{n}\times\frac{\Gamma\left(\frac{\frac{2}{n}-1+1}{2}\right)\Gamma\left(\frac{1-\frac{2}{n}+1}{2}\right)}{2\Gamma\left(\frac{\frac{2}{n}-1+1-\frac{2}{n}+2}{2}\right)}$$

$$= \frac{2}{n}\times\frac{\Gamma\left(\frac{2}{2n}\right)\Gamma\left(\frac{n-2+n}{2n}\right)}{2\Gamma\left(\frac{2}{2}\right)}$$

$$= \frac{2}{n}\times\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2n-2}{2n}\right)}{2\Gamma(1)}$$

$$\int_0^1 \frac{dx}{(1-x^n)^{1/n}} = \frac{1}{n}\times\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{n-1}{n}\right)}{1} \quad [\because \Gamma(1)=1]$$

$$= \frac{1}{n}\times\frac{\pi}{\sin\frac{\pi}{n}}$$

$$\left[\because \Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right) = \frac{\pi}{\sin\frac{\pi}{n}}\right]$$

$$= \frac{\pi}{n}\operatorname{cosec}\left(\frac{\pi}{n}\right)$$

$$\therefore \int_0^1 \frac{dx}{(1-x^n)^{1/n}} = \frac{\pi}{n}\operatorname{cosec}\left(\frac{\pi}{n}\right)$$

Q46. Evaluate $\int_0^\infty e^{-ax} x^{m-1} \sin bx\, dx$ interms of Gamma function.

Answer : Jan.-12, Q15(b)

Given integral is,

$$\int_0^\infty e^{-ax} x^{m-1} \sin bx\, dx$$

From the property of gamma function,

$$\int_0^\infty e^{-ax} x^{m-1}\, dx = \frac{\Gamma(m)}{a^m} \quad ...(1)$$

Substituting $a = a + ib$ in equation (1),

$$\int_0^\infty e^{-(a+ib)} x^{m-1} dx = \frac{\Gamma(m)}{(a+ib)^m}$$

Consider, ... (2)

$$e^{-(a+ib)x} = e^{-ax}.e^{-ibx}$$

$$\Rightarrow \quad e^{-(a+ib)x} = e^{-ax}(\cos bx - i\sin bx) \quad ...(3)$$

Substituting equation (3) in equation (2),

$$\int_0^\infty e^{-ax}(\cos bx - i\sin bx)x^{m-1} dx = \frac{\Gamma(m)}{(a+ib)^m} \quad ...(4)$$

Let,

$a = r\cos\theta$; $b = r\sin\theta$

$\Rightarrow \quad a^2 + b^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$\Rightarrow \quad a^2 + b^2 = r^2(\cos^2\theta + \sin^2\theta)$

$\Rightarrow \quad a^2 + b^2 = r^2$

And $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$(a + ib)^m = (r\cos\theta + ir\sin\theta)^m$

$= r^m(\cos\theta + i\sin\theta)^m$

$\therefore \quad (a + ib)^m = r^m(\cos m\theta + i\sin m\theta) \quad ...(5)$

Substituting equation (5) in equation (4),

$$\int_0^\infty e^{-ax}(\cos bx - i\sin bx)x^{m-1} dx = \frac{\Gamma(m)}{r^m(\cos m\theta + i\sin m\theta)}$$

$$= \frac{\Gamma(m)}{r^m}(\cos m\theta + i\sin m\theta)^{-1}$$

$$= \frac{\Gamma(m)}{r^m}(\cos m\theta - i\sin m\theta)$$

Equating real and imaginary parts,

$$\int_0^\infty e^{-ax} x^{m-1}\cos bx\, dx = \frac{\Gamma(m)}{r^m}\cos m\theta \quad \text{and}$$

$$\int_0^\infty e^{-ax} x^{m-1}\sin bx\, dx = \frac{\Gamma(m)}{r^m}\sin m\theta$$

$$\therefore \int_0^\infty e^{-ax} x^{m-1}\sin bx\, dx = \frac{\Gamma(m)}{r^m}\sin m\theta$$

Where,

$r^2 = a^2 + b^2$ and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

Q47. Evaluate $\int_0^{\frac{\pi}{2}} \sin^5\theta\cos^7\theta d\theta$ using Beta and Gamma functions.

Answer : June-15, Q14(a)

Given integral is,

$$\int_0^{\pi/2} \sin^5\theta\cos^7\theta\, d\theta$$

From the property of beta function,

$$\int_0^{\pi/2} \sin^m\theta\cos^n\theta\, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2},\frac{n+1}{2}\right)$$

Here, $m = 5$, $n = 7$

Substituting the corresponding values in above integral,

$$\int_0^{\pi/2} \sin^5\theta\cos^7\theta\, d\theta = \frac{1}{2}\beta\left(\frac{5+1}{2},\frac{7+1}{2}\right)$$

$$= \frac{1}{2}\beta\left(\frac{6}{2},\frac{8}{2}\right) = \frac{1}{2}\beta(3,4)$$

$$= \frac{1}{2}\frac{\Gamma(3)\Gamma(4)}{\Gamma(3+4)} \quad \left[\because \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right]$$

$$= \frac{1}{2}\frac{\Gamma(3)\Gamma(4)}{\Gamma(7)} = \frac{1}{2}\frac{(3-1)!(4-1)!}{(7-1)!}$$

$$[\because \Gamma(n) = (n-1)!]$$

$$= \frac{1}{2}\left(\frac{2!3!}{6!}\right) = \frac{1}{2}\left(\frac{2\times 6}{720}\right)$$

$$= \frac{1}{120}$$

$$\therefore \int_0^{\pi/2} \sin^5\theta\cos^7\theta\, d\theta = \frac{1}{120}$$

4.2 ERROR FUNCTIONS

Q48. Show that erf(x) + erfc(x) = 1

Answer :

The error function is defined as,

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt \qquad \text{... (1)}$$

The complementary error function is defined as,

$$erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt \qquad \text{... (2)}$$

Adding equations (1) and (2),

$$erf(x) + erfc(x) = \frac{2}{\sqrt{\pi}}\left[\int_0^x e^{-t^2}\,dt + \int_x^\infty e^{-t^2}\,dt\right]$$

$$= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\,dt$$

$$= \frac{2}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2} \qquad \left[\because \int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}\right]$$

$$= 1$$

$\therefore erf(x) + erfc(x) = 1$

Q49. Evaluate $\frac{d}{dx}[erf(ax)]$.

Answer : [Model Paper-1, Q14(b) | April-16, Q7]

The error function is given as,

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

Substituting $x = ax$ in above equation,

$$erf(ax) = \frac{2}{\sqrt{\pi}}\int_0^{ax} e^{-t^2}\,dt$$

Differentiating above equation with respect to 'x',

$$\frac{d}{dx}(erf(ax)) = \frac{d}{dx}\frac{2}{\sqrt{\pi}}\int_0^{ax} e^{-t^2}\,dt$$

$$\Rightarrow \quad \frac{d}{dx}(erf(ax)) = \frac{2}{\sqrt{\pi}}\frac{d}{dx}\int_0^{ax} e^{-t^2}\,dt$$

Applying rule of differentiation under integral sign,

$$\frac{d}{dx}(erf(ax)) = \frac{2}{\sqrt{\pi}}\left[\int_0^{ax}\frac{\partial}{\partial x}(e^{-t^2})\,dt + \frac{d}{dx}(ax)e^{-a^2x^2} - \frac{d}{dx}(0).1\right]$$

$$= \frac{2}{\sqrt{\pi}}[0 + ae^{-a^2x^2} - 0] = \frac{2ae^{-a^2x^2}}{\sqrt{\pi}}$$

$$\therefore \quad \frac{d}{dx}(erf(ax)) = \frac{2ae^{-a^2x^2}}{\sqrt{\pi}}$$

Q50. Show that $\int_0^t erf(\alpha x)dx = t\, erf(\alpha t) + \frac{1}{\alpha\sqrt{\pi}}[e^{-\alpha^2 t^2} - 1]$.

Answer : [Model Paper-3, Q14(b) | May/June-17, Q14(a)]

Given integral is,

$$I = \int_0^t erf(\alpha x)dx = \int_0^t 1\, erf(\alpha x)dx$$

$$= erf(\alpha x)\int_0^t 1dx - \int_0^t \left(\frac{d}{dx}(erf(\alpha x))\int 1dx\right)dx$$

$$= [x\, erf(\alpha x)]_0^t - \int_0^t x\left[\frac{d}{dx}(erf(\alpha x))\right]dx$$

$$= t\, erf(\alpha t) - \int_0^t x\frac{2\alpha}{\sqrt{\pi}}e^{-\alpha^2x^2}dx \qquad \left[\because \frac{d}{dx}(erf(\alpha x)) = \frac{2\alpha}{\sqrt{\pi}}e^{-\alpha^2x^2}\right]$$

$$\int_0^t erf(\alpha x)dx = t\, erf(\alpha t) - \frac{2\alpha}{\sqrt{\pi}}\int_0^t xe^{-\alpha^2x^2}dx \qquad \left[\because erf(\alpha x) = \frac{2}{\sqrt{\pi}}\int_0^t xe^{-t^2}dt\right] \qquad \text{... (1)}$$

Consider,

$$\int_0^t xe^{-\alpha^2x^2}dx$$

Let,

$y = \alpha^2x^2$

$dy = \alpha^2 2xdx$

$\Rightarrow \quad xdx = \frac{dy}{2\alpha^2}$

L.L : $x = 0 \Rightarrow y = 0$,

U.L : $x = t \Rightarrow y = \alpha^2t^2$

$$\therefore \quad \int_0^t xe^{-\alpha^2x^2}dx = \int_0^{\alpha^2t^2} e^{-y}\frac{dy}{2\alpha^2}$$

$$= \frac{1}{2\alpha^2}\int_0^{\alpha^2t^2} e^{-y}dy$$

$$= \frac{1}{2\alpha^2}[-e^{-y}]_0^{\alpha^2t^2}$$

$$= \frac{-1}{2\alpha^2}[e^{-\alpha^2t^2} - e^0]$$

$$= \frac{-1}{2\alpha^2}[e^{-\alpha^2t^2} - 1]$$

$$\therefore \quad \int_0^t xe^{-\alpha^2x^2}dx = \frac{-1}{2\alpha^2}[e^{-\alpha^2t^2} - 1] \qquad \text{... (2)}$$

Substituting equation (2) in equation (1),

$$\int_0^t erf(\alpha x)dx = t\, erf(\alpha t) - \frac{2\alpha}{\sqrt{\pi}}\left[\frac{-1}{2\alpha^2}\left[e^{-\alpha^2 t^2} - 1\right]\right]$$

$$= t\, erf(\alpha t) + \frac{1}{\alpha\sqrt{\pi}}\left[e^{-\alpha^2 t^2} - 1\right]$$

$$\therefore \quad \int_0^t erf(\alpha x)dx = t\, erf(\alpha t) + \frac{1}{\alpha\sqrt{\pi}}\left[e^{-\alpha^2 t^2} - 1\right].$$

Q51. Show that $\frac{d}{dx}\{erf\,(\alpha\, x)\} = \frac{2\alpha}{\sqrt{\pi}} e^{-\alpha^2 x^2}$.

Answer : [Model Paper-2, Q14(b) | Dec.-17, Q14(a)]

From the definition of error function,

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

$$\Rightarrow \quad erf(\alpha x) = \frac{2}{\sqrt{\pi}}\int_0^{\alpha x} e^{-t^2}dt$$

$$= \frac{2}{\sqrt{\pi}}\int_0^{\alpha x}\left(1 - \frac{t^2}{1!} + \frac{t^4}{2!} - \frac{t^6}{3!} + ...\right)dt$$

$$= \frac{2}{\sqrt{\pi}}\left[t - \frac{1}{1!}\frac{t^3}{3} + \frac{1}{2!}\frac{t^5}{5} - \frac{1}{3!}\frac{t^7}{7} + ...\right]_0^{\alpha x}$$

$$= \frac{2}{\sqrt{\pi}}\left[t - \frac{t^3}{3} + \frac{t^5}{10} + \frac{t^7}{42} + ...\right]_0^{\alpha x}$$

$$= \frac{2}{\sqrt{\pi}}\left[\left[\alpha x - \frac{(\alpha x)^3}{3} + \frac{(\alpha x)^5}{10} - \frac{(\alpha x)^7}{42} + ...\right] - \left[0 - \frac{0}{3} + \frac{0}{10} - \frac{0}{42} + ...\right]\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[\left[\alpha x - \frac{\alpha^3 x^3}{3} + \frac{\alpha^5 x^5}{10} - \frac{\alpha^7 x^7}{42} + ...\right] - 0\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[\alpha x - \frac{\alpha^3 x^3}{3} + \frac{\alpha^5 x^5}{10} - \frac{\alpha^7 x^7}{42} + ...\right]$$

$$= \frac{2\alpha}{\sqrt{\pi}}\left[x - \frac{\alpha^2 x^3}{3} + \frac{\alpha^4 x^5}{10} - \frac{\alpha^6 x^7}{42} + ...\right]$$

Differentiating on both sides with respect to x.

$$\frac{d}{dx}(erf(\alpha x)) = \frac{2\alpha}{\sqrt{\pi}}\left[\frac{d}{dx}\left(x - \frac{\alpha^2 x^3}{3} + \frac{\alpha^4 x^5}{10} - \frac{\alpha^6 x^7}{42} + ...\right)\right]$$

$$= \frac{2\alpha}{\sqrt{\pi}}\left[1 - \frac{\alpha^2}{3}3x^2 + \frac{\alpha^4}{10}5x^4 - \frac{\alpha^6}{42}7x^6 + ...\right]$$

$$= \frac{2\alpha}{\sqrt{\pi}}\left[1 - \alpha^2 x^2 + \frac{\alpha^4 x^4}{2} - \frac{\alpha^6 x^6}{6} + ...\right]$$

$$= \frac{2\alpha}{\sqrt{\pi}}\left[1 - \alpha^2 x^2 + \frac{(\alpha^2 x^2)^2}{2!} - \frac{(\alpha^2 x^2)^3}{3!} + ...\right]$$

$$= \frac{2\alpha}{\sqrt{\pi}}e^{-\alpha^2 x^2}$$

$$\therefore \frac{d}{dx}(erf(\alpha x)) = \frac{2\alpha}{\sqrt{\pi}}e^{-\alpha^2 x^2}.$$

4.3 POWER SERIES METHOD

Q52. Write about power series solution of a differential equation.

Answer :

If the equation $u_0(x)y'' + u_1(x)y' + u_2(x)y = 0$ has an ordinary or regular point $x = x_0$, then all the solutions of the equation are said to be analytic and have a power series expansion at $x = x_0$ of the form,

$$y(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + c_3(x - x_0)^3 + \ldots$$

or

$$\Rightarrow \quad y(x) = \sum_{m=0}^{\infty} c_m (x - x_0)^m$$

Where,

c_0, c_1, c_2 .... = Coefficients

x_0 = Centre of expansion of power series.

∴ The radius of convergence of the power series is,

$$R = \underset{m\to\infty}{Lt}\left|\frac{c_m}{c_{m+1}}\right|$$

or

$$R = \frac{1}{\underset{m\to\infty}{Lt}\sqrt[m]{|c_m|}}$$

Note

The series converges if the following conditions are satisfied,

(i) $|x - x_0| < R$

(ii) $\underset{m\to\infty}{Lt}\left|\frac{c_{m+1}}{c_m}\right| < 1$

(iii) The value of R is infinity.

Q53. Write the steps involved in finding the series solutions to differential equations around zero.

Answer :

Let the differential equation be,

$$P_0\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = 0 \qquad \text{... (1)}$$

Where,

P_0, P_1, P_2 - Polynomials in x.

And $P_0 \neq 0$ at $x = 0$

The sequence of steps involved in finding the series solutions at $x = 0$ are,

Step 1

Write the solution of equation (1) as,

$$y = c_0 + c_1 x + c_2 x^2 + ... + c_n x^n \qquad \text{.... (2)}$$

Step 2

Obtain the expressions for $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from equation (2).

Step 3

The values of 'y', $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are substituted in equation (1).

Step 4

Obtain the values of c_2, c_3, c_4 ... in terms of c_0 and c_1 by equating the coefficients of powers of x to zero.

Step 5

The required series solution is obtained by substituting the values of c_2, c_3, c_4 ... in equation (2).

Q54. Solve by series solution method of the equation $x\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+xy=0$, about x = 0

Answer : [Model Paper-2, Q17(a) | May/June-12, Q13]

Given differential equation is,

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=0 \quad \text{... (1)}$$

At $x = 0$, the coefficient of $\dfrac{d^2y}{dx^2} \neq 0$

Let the solution of equation (1) be,

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \ldots \quad \text{... (2)}$$

Differentiating equation (2), with respect to 'x'

$$\frac{dy}{dx} = 0 + c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \ldots \quad \text{... (3)}$$

Differentiating equation (3), with respect to 'x'

$$\frac{d^2y}{dx^2} = 2c_2 + 6c_3x + 12c_4x^2 + \ldots \quad \text{... (4)}$$

Substituting equations (2), (3) and (4) in equation (1),

$$x(2c_2 + 6c_3x + 12c_4x^2 + \ldots) + (c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \ldots) + x(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \ldots) = 0$$

$$\Rightarrow 2c_2x + 12c_3x^2 + 12c_4x^3 + c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + c_0x + c_1x^2 + c_2x^3 + c_3x^4 + c_4x^5 + \ldots = 0$$

Equating the coefficients of constant terms to zero,

$$c_1 = 0 \quad \text{... (5)}$$

Equating the coefficients of x terms to zero,

$$2c_2 + 2c_2 + c_0 = 0 \Rightarrow 4c_2 + c_0 = 0$$

$$\Rightarrow \quad c_2 = \frac{-c_0}{4} \quad \text{... (6)}$$

Equating the coefficients of x^2 term to zero,

$$12c_3 + 3c_3 + c_1 = 0 \Rightarrow 15c_3 = -c_1$$

$$\Rightarrow \quad c_3 = \frac{-c_1}{15} \quad \text{.... (7)}$$

$$\Rightarrow \quad c_3 = 0 \quad \text{(From equation (5))}$$

Equating the coefficients of x^3 term to zero,

$$12c_4 + 4c_4 + c_2 = 0 \Rightarrow 16c_4 + c_2 = 0$$

$$\Rightarrow \quad c_4 = \frac{-c_2}{16} \quad \text{... (8)}$$

Substituting equation (6) in equation (8),

$$c_4 = \frac{c_0}{64} \quad \text{... (9)}$$

Substituting equations (5), (6) and (9) in equation (2),

$$y = c_0 + 0(x) + \left(\frac{-c_0}{4}\right)x^2 + 0(x^3) + \left(\frac{c_0}{64}\right)x^4 + \ldots\ldots\ldots\ldots$$

$$= c_0 - \frac{c_0}{4}x^2 + \frac{c_0}{64}x^4 + \ldots\ldots\ldots\ldots$$

$$= c_0\left[1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 + \ldots\ldots\ldots\ldots\right]$$

$$\therefore \quad y = c_0\left[1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 + \ldots\ldots\ldots\ldots\right]$$ is the required power series solution.

Q55. Find the series solution about x = 0 of the differential equation (1 – x²)y'' – xy' + 2y = 0

Answer : June-13, Q14

Given differential equation is,

$$(1 - x^2)y'' - xy' + 2y = 0 \quad \text{... (1)}$$

At $x = 0$ the coefficient of $y'' \neq 0$

Let the solution of equation (1) be,

$$y = \sum_{m=0}^{\infty} c_m x^m$$

$$\Rightarrow \quad y = c_0 + c_1x + c_2x^2 + \ldots\ldots \quad \text{... (2)}$$

Differentiating equation (2) with respect to 'x',

$$y' = \sum_{m=1}^{\infty} m\, c_m x^{m-1}$$

$$\Rightarrow \quad y' = 1\, c_1 + 2\, c_2x + 3\, c_3x^2 + \ldots\ldots \quad \text{... (3)}$$

Differentiating equation (3) with respect to 'x',

$$y'' = \sum_{m=2}^{\infty} m\,(m-1)c_m x^{m-2}$$

$$\Rightarrow \quad y'' = 2\, c_2 + 6\, c_3x + 12\, c_4x^2 + \ldots\ldots \quad \text{... (4)}$$

Substituting equations (2), (3) and (4) in equation (1),

$$(1 - x^2)(2\, c_2 + 6\, c_3x + 12\, c_4x^2) - x(c_1 + 2c_2x + 3\, c_3x^2) + 2(c_0 + c_1x + c_2x^2) = 0$$

$$\Rightarrow \quad 2\, c_2 + 6\, c_3x + 12\, c_4x^2 - 2\, c_2x^2 - 6\, c_3x^3 - 12\, c_4x^4 - c_1x - 2\, c_2x^2 - 3\, c_3x^3 + 2\, c_0 + 2\, c_1x + 2\, c_2x^2 = 0$$

$$\Rightarrow \quad 2\, c_2 + 2\, c_0 + 6\, c_3x\, c_1x + 12\, c_4x^2 - 2\, c_2x^2 - 9\, c_3x^3 - 12\, c_4x^4 = 0$$

$$\Rightarrow \quad 2(c_2 + c_0) + (6\, c_3 + c_1)\, x + (12\, c_4 - 2\, c_2)x^2 - (9\, c_3)x^3 - (12\, c_4)x^4 = 0 \quad \text{... (5)}$$

Equating constants, coefficient of x, x^2, x^3, x^4, on both sides,

Constants	Coefficient of x	Coefficient of x^2	Coefficient of x^3	Coefficient of x^4
$2(c_2 + c_0) = 0$	$6c_3 + c_1 = 0$	$12c_4 - 2c_2 = 0$	$-9c_3 = 0$	$-12c_4 = 0$
$c_2 + c_0 = 0$	$6c_3 = -c_1$	$12c_4 = 2c_2$	$c_3 = 0$	$c_4 = 0$
$c_2 = -c_0$	$c_3 = \frac{-c_1}{6}$	$c_4 = \frac{c_2}{6}$		
		$\Rightarrow c_4 = -\frac{c_0}{6}$		

Substituting the values of c_2, c_3 and c_4 in equation (2),

$$y = c_0 + c_1x + (-c_0)x^2 + \ldots\ldots$$

$$\Rightarrow \quad y = c_0 - c_0x^2 + c_1x$$

$$\Rightarrow \quad y = c_0\left(1 - x^2\right) + c_1x$$

$\therefore \quad y = c_0\left(1 - x^2\right) + c_1x$ is the required power series.

Q56. Find the series solution about x = 0 of the equation (1 – x²)y'' – 2xy' + 6y = 0.

Answer : Jan.-12, Q14

Given differential equation is,

$$(1 - x^2)y'' - 2xy' + 6y = 0 \quad \text{... (1)}$$

Equation (1) can be written as,

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+6y = 0 \quad \text{... (2)}$$

At, $x = 0$, the coefficients of $\frac{d^2y}{dx^2} \neq 0$

Let the solution of equation (1) be,

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \ldots \quad \text{... (3)}$$

Differentiating equation (3), with respect to 'x'

$$\frac{dy}{dx} = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \ldots \quad \text{... (4)}$$

Differentiating equation (4), with respect to 'x'

$$\frac{d^2y}{dx^2} = 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \ldots \quad \text{... (5)}$$

Substituting equations (5), (4) and (3) in equation (2),

$$(1-x^2)(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \ldots) - 2x(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \ldots) + 6(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \ldots) = 0$$

$$\Rightarrow \quad [2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 - 2c_2x^2 - 6c_3x^3 - 12c_4x^4 + 20c_5x^5 + \ldots] + [-2c_1x - 4c_2x^2 - 6c_3x^3 - 8c_4x^4 - 10c_5x^5 + \ldots]$$

$$+ [6c_0 + 6c_1x + 6c_2x^2 + 6c_3x^3 + 6c_4x^4 + 6c_5x^5 + \ldots] = 0$$

Equating the constant term to zero,

$$2c_2 + 6c_0 = 0 \Rightarrow c_2 = \frac{-6c_0}{2} = 3c_0$$

$$\therefore \quad c_2 = -3c_0 \quad \text{.... (6)}$$

Equating the coefficients of x term to zero,

$$6c_3 - 2c_1 + 6c_1 = 0 \Rightarrow 6c_3 + 4c_1 = 0$$

$$\Rightarrow \quad c_3 = \frac{-4}{6}c_1 = \frac{2}{3}c_1$$

$$\therefore \quad c_3 = \frac{-2}{3}c_1 \quad \text{.... (7)}$$

Equating the coefficients of x^2 term to zero,

$$12c_4 - 4c_2 - 2c_2 + 6c_2 = 0 \Rightarrow 12c_4 = 0$$

$$\therefore \quad c_4 = 0 \quad \text{... (8)}$$

Equating the coefficients of x^3 term to zero,

$$20c_5 - 6c_3 - 6c_3 + 6c_3 = 0 \Rightarrow 20c_5 = 6c_3$$

$$\Rightarrow \quad c_5 = \frac{6c_3}{20} = \frac{3}{10}c_3$$

$$\therefore \quad c_5 = \frac{3c_3}{10} \quad \text{... (9)}$$

Substituting equation (7) in equation (9),

$$c_5 = \frac{3}{10}\left(\frac{-2}{3}c_1\right) = \frac{1}{5}c_1$$

$$\therefore \quad c_5 = \frac{-1}{5}c_1 \quad \text{... (10)}$$

Substituting equations (6), (7), (8) and (10) in equation (3),

$$y = c_0 + c_1x - (3c_0)\,x^2 - \left(\frac{2}{3}c_1\right)x^3 + 0 - \frac{1}{5}c_1\ x^5 + ...$$

$$= c_0\left[1-3x^2+.....\right]+c_1\left[x-\frac{2}{3}x^3-\frac{1}{5}x^5+.....\right]$$

$\therefore\quad y = c_0\left[1-3x^2+.....\right]+c_1\left[x-\frac{2}{3}x^3-\frac{1}{5}x^5+.....\right]$ is the required power series solution.

Q57. Obtain the series solution of the equation.

$$\mathbf{x(1-x)\frac{d^2y}{dx^2}-(1+3x)\frac{dy}{dx}-y=0\ \ about\ x=0}$$

Answer : Dec.-13, Q13

Given differential equation is,

$$x(1-x)\frac{d^2y}{dx^2}-(1+3x)\frac{dy}{dx}-y=0 \quad ... (1)$$

At $x = 0$, The coefficients of $\frac{d^2y}{dx^2} \neq 0$

Let the solution of equation (1) be,

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \ldots \quad ... (2)$$

Differentiating equation (2), with respect to 'x',

$$\frac{dy}{dx} = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \ldots \quad ... (3)$$

Differentiating equation (3) with respect to 'x',

$$\frac{d^2y}{dx^2} = 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \ldots \quad ... (4)$$

Substituting equations (4), (3) and (2) in equation (1),

$\Rightarrow$ $x(1-x)(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \ldots) - (1 + 3x)(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \ldots) - (c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \ldots) = 0$

$\Rightarrow$ $(x - x^2)(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \ldots) - (1 + 3x)(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \ldots) - c_0 - c_1x - c_2x^2 - c_3x^3 - c_4x^4 - c_5x^5 - \ldots = 0$

$\Rightarrow$ $2c_2x + 6c_3x^2 + 12c_4x^3 + 20c_5x^4 - 2c_2x^2 - 6c_3x^3 - 12c_4x^4 - 20c_5x^5 - c_1 - 2c_2x - 3\,c_3x^2 - 4c_4x^3 - 5c_5x^4 - 3c_1x - 6c_2\,x^2 - 9c_3x^3 - 12c_4x^4 - 15c_5x^5 - c_0 - c_1x - c_2x^2 - c_3x^3 - c_4x^4 - c_5x^5\ ..... = 0$

Equating the constant term to zero,

$$- c_1 - c_0 = 0$$

$$\Rightarrow \quad c_0 = - c_1 \quad .... (5)$$

Equating the coefficients of x term to zero,

$$2c_2 - 2c_2 - 3c_1 - c_1 = 0 \Rightarrow -4c_1 = 0$$

$$\Rightarrow \quad c_1 = 0 \quad ... (6)$$

Equating the coefficients of x^2 term to zero,

$$6c_3 - 2c_2 \quad -3c_3 - 6c_2 - c_2 = 0 \Rightarrow 3c_3 - 9c_2 = 0$$

$$\Rightarrow \quad 3c_3 = 9c_2$$

$$\Rightarrow \quad c_2 = \frac{1}{3}c_3 \quad ... (7)$$

Equating the coefficients of x^3 terms to zero,

$12c_4 - 6c_3 - 20c_3 - 4c_4 - 9c_3 - c_3 = 0$

$\Rightarrow \quad 8c_4 - 36c_3 \quad = 0$

$\Rightarrow \quad c_4 = \frac{36}{8}c_3$

$\Rightarrow \quad c_4 = 4c_3$ ... (8)

Equating the coefficients of x^4 term to zero,

$20c_5 - 12c_4 - 5c_5 - 12c_4 - c_4 = 0 \Rightarrow 15c_5 - 25c_4 = 0$

$c_5 = \frac{25}{15}c_4 = \frac{5}{3}c_4$

$\therefore \quad c_5 = \frac{5}{3}c_4$ ... (9)

Substituting equation (6) in equation (5),

$c_0 = 0$ ... (10)

Substituting equation (9) in equation (8),

$c_5 = \frac{5}{3} \times 4c_3$

$\Rightarrow \quad c_5 = \frac{20}{3}c_3$ ... (11)

Substituting equations (5), (6), (7), (10), (11) in equation (2),

$$y = 0 + 0(x) + \frac{1}{3}c_3x^2 + 4c_3x^3 + \frac{20}{3}c_3x^4 + ...$$

$$= c_3\left[\frac{1}{3}x^2 + 4x^3 + \frac{20}{3}x^4 + ...\right]$$

$\therefore \quad y = c_3\left[\frac{1}{3}x^2 + 4x^3 + \frac{20}{3}x^4 + ...\right]$ is the required power series solution.

4.4 LEGENDER'S DIFFERENTIAL EQUATIONS AND LEGENDER'S POLYNOMIAL P_n(x), RODRIGUE'S FORMULA (WITHOUT PROOF)

Q58. What is Legendre's differential equation? Write its general solutions.

Answer :

The Legendre's differential equation is given as,

$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ ... (1)

Where,

n – Real number.

Equation (1) can also be expressed as,

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0$$

General Solutions of Legendre's Differential Equation

The Legendre's differential equation has two general solutions

i.e., For $m = n$ and $m = -n$

Case (i)

The general solution of Legendre's equation for $m = n$ is,

$$y = a_0\left[x^n - \frac{n(n-1)}{(2n-1)2}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3)2.4}x^{n-4}.....\right]$$

Case (ii)

The general solution of Legendre's equation for $m = -n$ is,

$$y = a_0\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)}x^{-n-5} + ...\right]$$

Q59. Write about the following,

(i) Legendre's function of the first kind ($P_n(x)$)

(ii) Legendre's function of the second kind ($Q_n(x)$).

Answer :

(i) Legendre's Function of the First Kind ($P_n(x)$)

The Legendre's differential equation is given as,

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\,y = 0 \qquad ... (1)$$

One of the solution of equation (1) is,

$$y = P_n(x) = a_0\left[x^n - \frac{n(n-1)}{(2n-1)2}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3)2.4}x^{n-4}......\right] \qquad ... (2)$$

Where,

a_0 - Arbitrary constant.

Moreover, when n is a positive integer,

$$a_0 = \frac{1.3.5...(2n-1)}{n!} \qquad ... (3)$$

Substituting equation (3) in equation (2)

$$P_n(x) = \frac{1.3.5...(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{(2n-1)2}x^{n-2} + ...\right]$$

This function $P_n(x)$ is known as the Legendre's function of the first kind.

(ii) Legendre's Function of the Second Kind ($Q_n(x)$)

The Legendre's differential equation is given as,

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

Another solution of equation (1) is,

$$Q_n(x) = y = a_0\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + ...\right] \qquad ... (4)$$

Where,

$$a_0 = \frac{n!}{1.3.5...(2n+1)} \qquad ... (5)$$

Substituting equation (5) in equation (4),

$$Q_n(x) = y = \frac{n!}{1.3.5...(2n+1)}\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + ...\right]$$

Note

$P_n(x)$ is a terminating series and $Q_n(x)$ is a non-terminating series.

Q60. Express the following in terms of Legendre polynomials $4x^3 - 2x^2 - 3x + 8$.

Answer :

Given function is,

$$f(x) = 4x^3 - 2x^2 - 3x + 8 \qquad ... (1)$$

From Rodrigue's formula,

$$P_n(x) = \frac{1}{2^n n!}.\frac{d^n}{dx^n}(x^2-1)^n$$

$$P_3(x) = \frac{1}{2}(5x^3-3x) \quad \text{... (2)}$$

$$P_2(x) = \frac{1}{2}(3x^2-1) \quad \text{... (3)}$$

$$P_1(x) = x \quad \text{... (4)}$$

and $P_0(x) = 1$ ... (5)

From equation (2),

$$5x^3 - 3x = 2P_3(x)$$

$$\Rightarrow \quad x^3 = \frac{2}{5}P_3(x)+\frac{3}{5}(x)$$

$$\Rightarrow \quad x^3 = \frac{2}{5}P_3(x)+\frac{3}{5}P_1(x) \quad [\because \text{From equation (4)}] \quad \text{... (6)}$$

From equation (3),

$$3x^2 - 1 = 2P_2(x)$$

$$\Rightarrow \quad x^2 = \frac{2}{3}P_2(x)+\frac{1}{3}(1)$$

$$\Rightarrow \quad x^2 = \frac{2}{3}P_2(x)+\frac{1}{3}P_0(x) \quad [\because \text{From equation (5)}] \quad \text{... (7)}$$

Substituting equations (4), (5) (6) and (7) in equation (1),

$$f(x)= 4\left[\frac{2}{5}P_3(x)+\frac{3}{5}P_1(x)\right]-2\left[\frac{2}{3}P_2(x)+\frac{1}{3}P_0(x)\right]-3P_1(x)+8$$

$$\Rightarrow \quad f(x)= \frac{8}{5}P_3(x)+\frac{12}{5}P_1(x)-\frac{4}{3}P_2(x)-\frac{2}{3}P_0(x)-3P_1(x)+8P_0(x) \quad [\because P_0(x) = 1]$$

$$= \frac{8}{5}P_3(x)-\frac{4}{3}P_2(x)+\left[\frac{12}{5}-3\right]P_1(x)+\left[8-\frac{2}{3}\right]P_0(x)$$

$$= \frac{8}{5}P_3(x)-\frac{4}{5}P_2(x)-\frac{3}{5}P_1(x)+\frac{22}{3}P_0(x)$$

$$\therefore \; f(x) = \frac{8}{5}P_3(x)-\frac{4}{3}P_2(x)-\frac{3}{5}P_1(x)+\frac{22}{3}P_0(x)$$

Q61. Using Rodrigue's formula obtain the values of $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$, $P_4(x)$ respectively.

Answer : **Model Paper-3, Q17(a)**

From Rodrigue's formula,

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \quad \text{... (1)}$$

$P_0(x)$

Substituting $n = 0$, in equation (1),

$$P_0(x) = \frac{1}{2^0 0!}\frac{d^0}{dx^0}(x^2-1)^0 = 1$$

$$\therefore \; P_0(x) = 1$$

$P_1(x)$

Substituting $n = 1$, in equation (1),

$$P_1(x) = \frac{1}{2^1 1!}\frac{d}{dx}(x^2-1)^1$$

$$\Rightarrow \quad P_1(x) = \frac{2x}{2} = x$$

$$\therefore P_1(x) = x$$

$P_2(x)$

Substituting $n = 2$ in equation (1),

$$P_2(x) = \frac{1}{2^2.2!}.\frac{d^2}{dx^2}(x^2-1)^2$$

$$= \frac{1}{4.2}.\frac{d^2}{dx^2}(x^2-1)^2$$

$$= \frac{1}{8}.\frac{d^2}{dx^2}(x^4+1-2x^2)$$

$$= \frac{1}{8}\frac{d}{dx}\left[\frac{d}{dx}(x^4)+\frac{d}{dx}(1)-2\frac{d}{dx}(x^2)\right]$$

$$= \frac{1}{8}.\frac{d}{dx}(4x^3+0-4x)$$

$$= \frac{1}{8}.4\frac{d}{dx}(x^3-x) = \frac{1}{2}.\frac{d}{dx}(x^3-x)$$

$$= \frac{1}{2}(3x^2-1)$$

$$\therefore P_2(x) = \frac{1}{2}(3x^2-1)$$

$P_3(x)$

Substituting $n = 3$ in equation (1),

$$P_3(x) = \frac{1}{2^3 3!}\frac{d^3}{dx^3}(x^2-1)^3$$

$$= \frac{1}{48}\frac{d^3}{dx^3}[x^6-3x^4+3x^2-1]$$

$$= \frac{1}{48}\frac{d^2}{dx^2}(6x^5-12x^3+6x)$$

$$= \frac{1}{48}\frac{d}{dx}(30x^4-36x^2+6)$$

$$= \frac{1}{48}(120x^3-72x)$$

$$= \frac{1}{2}(5x^3-3x)$$

$$\therefore P_3(x) = \frac{1}{2}(5x^3-3x)$$

$P_4(x)$

Substituting $n = 4$, in equation (1),

$$P_4(x) = \frac{1}{2^4.4!}.\frac{d^4}{dx^4}(x^2-1)^4$$

$$= \frac{1}{16\times 24}.\frac{d^4}{dx^4}[(x^2-1)^2]^2$$

$$= \frac{1}{384}.\frac{d^4}{dx^4}(x^4+1-2x^2)^2$$

$$= \frac{1}{384}.\frac{d^4}{dx^4}(x^8+1+2x^4+4x^4-4x^2-4x^6)$$

$$= \frac{1}{384}.\frac{d^4}{dx^4}(x^8-4x^6+6x^4-4x^2+1)$$

$$= \frac{1}{134}\frac{d^3}{dx^3}\left(8x^7-24x^5+24x^3-8x\right)$$

$$= \frac{1}{384}\frac{d^2}{dx^2}(56x^6-120x^4+72x^2-8)$$

$$= \frac{1}{384}.\frac{d}{dx}(336x^5-480x^3+144x)$$

$$= \frac{1}{384}\left(1680x^4-1440x^2+144\right)$$

$$= \frac{48}{384}[35x^4-30x^2+3]$$

$$= \frac{1}{8}\left[35x^4-30x^2+3\right]$$

$$\therefore P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$$

Q62. Using Rodrigue's formula prove that $\int_{-1}^{1} x^m P_n(x)dx = 0$ if m < n.

Answer :

Given that,

$$\int_{-1}^{1} x^m P_n(x)\ dx = 0 \text{ for } m < n$$

Consider,

$$\int_{-1}^{1} x^m P_n(x)dx$$

From Rodrigue's formula,

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$\therefore \int_{-1}^{1} x^m P_n(x)dx = \int_{-1}^{1} x^m \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n dx$$

$$= \frac{1}{2^n n!} \int_{-1}^{1} x^m \frac{d^n}{dx^n}(x^2-1)^n \, dx$$

$$= \frac{1}{2^n n!} \int_{-1}^{1} x^m \frac{d}{dx}\left[\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right] dx$$

$$= \frac{1}{2^n n!}\left[\left[x^m \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right]_{-1}^{1} - \int_{-1}^{1} mx^{m-1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx\right] \quad \text{[Integration by parts]}$$

$$= 0 - \frac{m}{2^n n!} \int_{-1}^{1} x^{m-1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx$$

$$\Rightarrow \int_{-1}^{1} x^m P_n(x)dx = \frac{(-1)m}{2^n n!} \int_{-1}^{1} x^{m-1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n dx$$

Integrating above equation with respect to x,

$$\int_{-1}^{1} x^m P_n(x)dx = \frac{(-1)^2 m(m-1)}{2^n n!} \int_{-1}^{1} x^{m-2} \frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n dx$$

Integrating the above equation m times,

$$\int_{-1}^{1} x^m P_n(x) = \frac{(-1)^m m}{2^n n!} \int_{-1}^{1} \frac{d^{n-m}}{dx^{n-m}}(x^2-1)^n dx$$

$$= \left[\frac{(-1)^m m!}{2^n n!} \frac{d^{n-m-1}}{dx^{n-m-1}}(x^2-1)^n\right]_{-1}^{1} \qquad \left[\because \left[\frac{d^{n-1}}{dx^{n-1}}(x^2-1)\right]_{-1}^{1} = 0\right]$$

$$= 0$$

$$\therefore \int_{-1}^{1} x^m P_n(x)dx = 0 \text{ for } m < n$$

Q63. Show that, $x^4 = \frac{1}{35}[8P_4(x) + 20P_2(x) + 7P_0(x)]$.

OR

Show that $x^4 = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)$.

Answer :

Given that,

$$x^4 = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)$$

From Rodrigue's formula,

$$P_4(x) = \frac{1}{8}\left[35x^4 - 30x^2 + 3\right] \quad \text{... (1)}$$

$$P_2(x) = \frac{1}{2}(3x^2-1) \quad \text{... (2)}$$

$$P_0(x) = 1 \quad \text{... (3)}$$

From equation (1),

$$8P_4(x) = 35x^4 - 30x^2 + 3$$

$$\Rightarrow \quad 35x^4 = 8P_4(x) + 30x^2 - 3 \qquad \text{... (4)}$$

From equation (2),

$$2P_2(x) = 3x^2 - 1$$

$$\Rightarrow \quad 3x^2 = 2P_2(x) + 1$$

$$\Rightarrow \quad x^2 = \frac{2P_2(x)+1}{3}$$

$$\Rightarrow \quad x^2 = \frac{2P_2(x)+P_0(x)}{3} \qquad [\because \text{ From equation (3)}] \qquad \text{... (5)}$$

Substituting equations (3) and (5) in equation (4),

$$35x^4 = 8P_4(x) + \frac{30[2P_2(x)+P_0(x)]}{3} - 3P_0(x)$$

$$\Rightarrow \quad 35x^4 = 8P_4(x) + 20P_2(x) + 10P_0(x) - 3P_0(x)$$

$$\Rightarrow \quad 35x^4 = 8P_4(x) + 20P_2(x) + 7P_0(x)$$

$$\Rightarrow \quad x^4 = \frac{1}{35}[8P_4(x) + 20P_2(x) + 7P_0(x)]$$

$$= \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)$$

$$\therefore \quad x^4 = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)$$

Q64. Prove that $\frac{1}{\sqrt{1-2xt+t^2}} =, \sum_{n=0}^{\infty} P_n(x)t^n \quad t \neq 1$

Answer : June-10, Q15(a)

Given that,

$$\frac{1}{\sqrt{1-2xt+t^2}} = P_0(x) + P_1(x)t + P^2(x)t^2 + ...$$

$$\frac{1}{\sqrt{1-2xt+t^2}} = \frac{1}{(1-2xt+t^2)^{1/2}}$$

$$= (1 - 2xt + t^2)^{-1/2}$$

$$= \left[1-t(2x-t)\right]^{-1/2} \qquad \text{... (1)}$$

From Binomial theorem,

$$(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}.\frac{3}{2}}{2!}x + \frac{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}}{3!}x^3 + ...$$

$$(1-x)^{-1/2} = 1 + \frac{2!}{(1!)^2.2^2}x + \frac{4!}{(2!)^2.2^4}x^2 + \frac{6!}{(3!)^2.2^6}x^3 + ...$$

$\therefore$ Equation (1) becomes,

$$\frac{1}{\sqrt{1-2xt+t^2}} = \left[1-t(2x-t)\right]^{-1/2}$$

$$= 1 + \frac{2!}{(1!)^2.2^2} \times t(2x-t) + \frac{4!}{(2!)^2.2^4} t^2(2x-t)^2 + ... \frac{(2n-2r)!}{((n-r)!)^2 2^{2n-2r}} t^{n-r}(2x-t)^{n-r}$$

$$+ \frac{(2n)!}{(n!)^2 2^{2n}} t^n (2x-t)^n + ...$$

From the above expansion, consider the term $\dfrac{(2n-2r)!}{((n-r)!)^2\, 2^{2n-2r}}\, t^{n-r}(2x-t)^{n-r}$

$$= \frac{(2n-2r)!}{((n-r)!)^2\, 2^{2n-2r}} \times t^{n-r} \times {}^{(n-r)}C_r\, (-t)^r\, (2x)^{n-r-r} \qquad [\because (x+y)^r = {}^nC_r\, y^r\, (x)^{n-r}\,]$$

$$= \frac{(2n-2r)!}{((n-r)!)^2\, 2^{2n-2r}}\; t^{n-r}\; {}^{(n-r)}C_r\; (2x)^{n-2r}$$

$$= \frac{(-1)^r (2n-2r)!}{((n-r)!)^2\, 2^{2n-2r}} \times t^{n-r+r} \frac{(n-r)!}{r!(n-r-r)!}\, (2x)^{n-2r} \qquad \left(\because {}^nC_r = \frac{n!}{r!(n-r)!}\right)$$

$$= \frac{(-1)^r (2n-2r)!}{((n-r)!)^2\, 2^{2n-2r}} \times t^n \frac{(n-r)!}{r!(n-2r)!}\, 2^{n-2r}\, x^{n-2r}$$

$$= \frac{(-1)^r (2n-2r)!}{(n-r)!\,r!(n-2r)!} \times t^n \times 2^{n-2r-(2n-2r)} \times x^{n-2r}$$

$$= \frac{(-1)^r (2n-2r)!}{r!(n-r)!\,(n-2r)!} \times t^n \times 2^{n-2n} \times x^{n-2r}$$

$$= \frac{(-1)^r (2n-2r)!}{r!(n-r)!\,(n-2r)!} \times t^n \times 2^{-n} \times x^{n-2r}$$

$$= \frac{(-1)^r (2n-2r)!}{2^n\, r!\,(n-r)!(n-2r)!} \times x^{n-2r} \times t^n$$

Considering every term of t^n,

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{r=0}^{\infty} \frac{(-1)^r\, (2n-2r)!}{2^n\, r!(n-r)!(n-2r)!} \times x^{n-2r} \times t^n$$

$$= \sum_{n=0}^{\infty} P_n(x).t^n \qquad \left(\because P_n(x) = \frac{(-1)^r (2n-2r)!}{2^n r!(n-r)!\,(n-2r)!}.x^{n-2r}\right)$$

$$= P_0(x).t^0 + P_1(x).t^1 + P_2(x).t^2 + \ldots$$

$$\therefore \frac{1}{\sqrt{1+2xt+t^2}} = P_0(x) + P_1(x).t + P_2(x).t^2 + \ldots.$$

Q65. Show that $P_n(-x) = (-1)^n P_n(x)$.

Answer : **Model Paper-1, Q17(a)**

Given that,

$P_n(-x) = (-1)^n P_n(x)$

From the generating function of Legendre's polynomial,

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x) \qquad \ldots (1)$$

Replacing x by $-x$ in equation (1),

$$\frac{1}{\sqrt{1-2(-x)t+t^2}} = \sum_{n=0}^{\infty} t^n P_n(-x)$$

$$\Rightarrow \frac{1}{\sqrt{1+2xt+t^2}} = \sum_{n=0}^{\infty} t^n P_n(-x) \qquad \ldots (2)$$

Replacing t by '$-t$' in equation (1),

$$\frac{1}{\sqrt{1-2x(-t)+(-t^2)}} = \sum_{n=0}^{\infty}(-t^n)P_n(x)$$

$$\Rightarrow \quad \frac{1}{\sqrt{1+2xt+t^2}} = \sum_{n=0}^{\infty}(-1)^n(t)^n P_n(x) \quad \text{... (3)}$$

From equations (2) and (3),

$$\sum_{n=0}^{\infty} t^n P_n(-x) = \sum_{n=0}^{\infty}(-1)^n(t)^n P_n(x)$$

$$\Rightarrow \quad P_n(-x) = (-1)^n P_n(x)$$

$$\therefore P_n(-x) = (-1)^n P_n(x)$$

Q66. Prove $\dfrac{1-z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)z^n P_n(x)$

Answer :

Given that,

$$\frac{1-z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)z^n P_n(x) \quad \text{... (1)}$$

From the generating function of Legendre's polynomial,

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x) \quad \text{... (2)}$$

Differentiating equation (2) with respect to 'z',

$$\frac{-1}{2}(1-2xz+z^2)^{-3/2}(-2x+2z) = \sum_{n=0}^{\infty} n.z^{n-1}.P_n(x)$$

$$\Rightarrow \quad \frac{x-z}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} n.z^{n-1}.P_n(x) \quad \text{... (3)}$$

Multiplying equation (3) with $2z$ on both sides,

$$\frac{2z(x-z)}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} 2z.nz^{n-1}.P_n(x)$$

$$\Rightarrow \quad \frac{2xz-2z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} 2nz^n P_n(x) \quad \text{... (4)}$$

Adding equations (2) and (4),

$$(1-2xz+z^2)^{-1/2} + \frac{2xz-2z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} z^n P_n(x) + \sum_{n=0}^{\infty} 2nz^n P_n(x)$$

$$\Rightarrow \quad \frac{(1-2xz+z)^{\frac{-1}{2}+\frac{3}{2}} + 2xz - 2z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} z^n P_n(x)(2n+1)$$

$$\Rightarrow \quad \frac{1-2xz+z^2+2xz-2z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)z^n P_n(x)$$

$$\therefore \quad \frac{1-z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty}(2n+1)z^n P_n(x)$$

UNIT 5

LAPLACE TRANSFORMS

PART-A

SHORT QUESTIONS WITH SOLUTIONS

Q1. Define the Laplace transform and state its domain and Kernel.

Answer :

Laplace Transform

Laplace transform is a mathematical tool that transforms a time-domain function to a frequency domain function and vice-versa.

Definition

Laplace transform of a function $f(t)$ is a linear integral transform which is defined as,

$$\bar{f}(s) = L\{f(t)\}$$

$$= \int_0^\infty f(t).e^{-st}.dt$$

Where,

$f(t)$ – Time-domain function

e^{-st} – Kernel of the transform $[k(s, t)]$

L – Laplace transform operator

s – Complex variable = $\sigma + j\omega$

$\bar{f}(s)$ – Complex variable function.

Laplace transform helps in better understanding of both time-domain and frequency domain functions and their properties.

Q2. Show that t^2 is of exponential order.

Answer : Model Paper-1, Q9

Given expression is,

$$f(t) = t^2$$

$$\Rightarrow \quad \underset{t\to\infty}{Lt}\, e^{-st}t^2 = \underset{t\to\infty}{Lt}\, \frac{t^2}{e^{st}}$$

$$= \frac{\infty}{\infty}$$

Applying L'Hospital rule.

$$= \underset{t\to\infty}{Lt}\, \frac{2t}{se^{st}} = \frac{\infty}{\infty}$$

$$= \underset{t\to\infty}{Lt}\, \frac{2}{s^2e^{st}}$$

$$= \frac{2}{\infty} = 0$$

$\therefore$ t^2 is of exponential order.

Q3. Show that the function f(t) = t² is of exponential order 3.

Answer :

Given function is,

$f(t) = t^2$

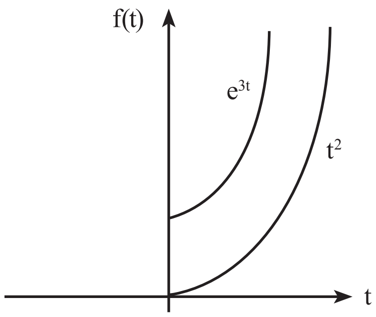
A function $f(t)$ is said to be exponential order function if for all $t \geq 0$ and for some constants M, k and K it satisfies, the condition,

$|f(t)| \leq M e^{kt}, t > K$

(or)

$\lim_{t \to \infty} |f(t)\ e^{-kt}|$ is finite.

The graph of e^{3t} and t^2 for $t \geq 0$ are shown in the figure below.



Figure

From the figure.

$|t^2| \leq e^{3t}$

$\therefore$ t^2 is an exponential function of order 3.

Q4. List out the various properties of Laplace transform and inverse laplace transfrom.

Answer :

Properties of Laplace Transform

(i) Linearity Property

If $L[f(t)] = \overline{f}(s)$ and $L[g(t)] = \overline{g}(s)$, then

$L[c_1 f(t) + c_2 g(t)] = c_1 L[f(t)] + c_2 L[g(t)]$

$= c_1 \overline{f}(s) + c_2 \overline{g}(s)$

Where, c_1 and c_2 are constants

(ii) $L[k] = \frac{k}{s}$ $(s > 0)$ and k is a constant

(iii) $L[t] = \frac{1}{s^2}$

(iv) $L[t^n] = \frac{n!}{s^{n+1}}$, where n is a positive integer

(v) $L[e^{at}] = \frac{1}{s-a}$, where $(s - a) > 0$

(vi) $L[\sin at] = \frac{a}{s^2+a^2}$ (if $s > 0$)

(vii) $L[\cos at] = \frac{s}{s^2+a^2}$ (if $s > 0$)

(viii) $L[\sin h\ at] = \frac{a}{s^2-a^2}$ (if $s > |a|$)

(ix) $L[\cos h\ at] = \frac{s}{s^2-a^2}$ (if $s > |a|$)

(x) $L[e^{at}.t^n] = \frac{n!}{(s-a)^{n+1}}$

(xi) $L[e^{at} \sin bt] = \frac{b}{(s-a)^2+b^2}$

(xii) $L[e^{at} \cos bt] = \frac{s-a}{(s-a)^2+b^2}$

(xiii) $L[e^{at} \sinh bt] = \frac{b}{(s-a)^2-b^2}$

(xiv) $L[e^{at} \cosh bt] = \frac{s-a}{(s-a)^2-b^2}$

(xv) $L[f(at)] = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

Properties of Inverse Laplace Transform

(i) Linearity Property

$L^{-1}\{a F_1(s) + b F_2(s)\} = a f_1(t) + b f_2(t)$

(ii) Shifting Property

If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(s - a)\} = e^{at} f(t)$

(iii) If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(t)\ dt$

(iv) If $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{e^{-as} F(s)\} = u(t - a) f(t - a)$

Q5. Find the Laplace transform of t³ + 5 cos t

Answer :

Given function is,

$f(t) = t^3 + 5 \cos t$

Taking Laplace transform on both sides,

$L\{f(t)\} = L\{t^3 + 5 \cos t\}$

$= L\{t^3\} + L\{5 \cos t\}$

$= \frac{3!}{s^{3+1}} + 5L\{\cos t\}$ $\left[\because L\{t^n\} = \frac{n!}{s^{n+1}}\right]$

$= \frac{6}{s^4} + 5\left(\frac{s}{s^2+1^2}\right)$ $\left(\because L\{\cos at\} = \frac{s}{s^2+a^2}\right)$

$= \frac{6}{s^4} + \frac{5s}{s^2+1}$

$\therefore L\{t^3 + 5\cos t\} = \frac{6}{s^4} + \frac{5s}{s^2+1}$

Q6. Find Laplace transform of $1 + 2\sqrt{t} + 3\sqrt{t}$.

Answer : Model Paper-1, Q10

Given that,

$1 + 2\sqrt{t} + 3\sqrt{t}$

Applying Laplace transform to the above expression,

$$L\left[1 + 2\sqrt{t} + 3\sqrt{t}\right] = L[1] + 2L[\sqrt{t}] + 3L[\sqrt{t}]$$

$$= \frac{1}{s} + 2L[t^{1/2}] + 3L[t^{1/2}]$$

$$= \frac{1}{s} + 2.\frac{\frac{1}{2}\Gamma\frac{1}{2}}{s^{\frac{1}{2}+1}} + 3.\frac{\frac{1}{2}\Gamma\frac{1}{2}}{s^{\frac{1}{2}+1}}$$

$$= \frac{1}{s} + \frac{\Gamma\frac{1}{2}}{s^{\frac{3}{2}}} + \frac{3}{2}\frac{\Gamma\frac{1}{2}}{s^{\frac{3}{2}}}$$

$$= \frac{1}{s} + \frac{\sqrt{\pi}}{s^{3/2}} + \frac{3}{2}\frac{\sqrt{\pi}}{s^{3/2}}$$

$$= \frac{1}{s} + \frac{5\sqrt{\pi}}{2s^{3/2}} \quad \left[\because \Gamma\frac{1}{2} = \sqrt{\pi}\right]$$

$$\therefore \; L[1+2\sqrt{t}+3\sqrt{t}\,] = \frac{1}{s} + \frac{5\sqrt{\pi}}{2s^{3/2}}$$

Q7. Find the Laplace transform of sin 2t sin 3t.

Answer :

Given function is, $\sin 2t \sin 3t$

$$\Rightarrow \quad \sin 2t \sin 3t = \frac{1}{2}(2\sin 2t + \sin 3t)$$

$$\Rightarrow \quad \sin 2t \sin 3t = \frac{1}{2}(\cos(2t-3t) - \cos(2t+3t))$$

$$[\because 2\sin A \sin B = \cos(A-B) - \cos(A+B)]$$

$$= \frac{1}{2}(\cos(-t) - \cos(5t))$$

$$= \frac{1}{2}(\cos(t) - \cos 5(t)) \quad \ldots (1)$$

Applying Laplace transform on both sides of equation (1),

$$L(\sin 2t \sin 3t) = \frac{1}{2}L(\cos(t) - \cos(5t))$$

$$= \frac{1}{2}[L(\cos(t)) - L(\cos(5t))]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+1^2} - \frac{s}{s^2+5^2}\right]$$

$$\left[\because \cos at = \frac{s}{s^2+a^2}\right]$$

$$= \frac{s}{2}\left[\frac{s^2+5^2-s^2-1^2}{(s^2+1^2)(s^2+5^2)}\right]$$

$$= \frac{s}{2}\left[\frac{25-1}{(s^2+1^2)(s^2+5^2)}\right]$$

$$= \frac{s}{2}\left[\frac{24}{(s^2+1^2)(s^2+5^2)}\right]$$

$$\therefore L[\sin 2t \sin 3t] = \frac{12s}{(s^2+1^2)(s^2+5^2)}$$

Q8. Define unit step function.

Answer :

The unit step function or Heaviside unit step function is defined as,

$$u(t-a) \text{ or } H(t-a) = \begin{cases} 0 & , t < a \\ 1 & , t > a \end{cases}$$

Where, $a > 0$

The graphical representation of unit step function is illustrated in figure below,

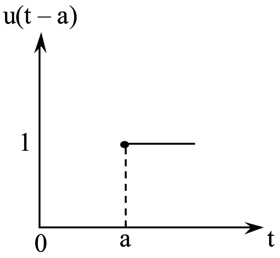


Figure: Unit Step Function

Q9. Define unit impulse function.

Answer :

The Dirac's delta function or unit impulse function can be defined as the limiting form of,

$$f_\varepsilon(t-a) = \begin{cases} 0 & , t < a \\ \frac{1}{\varepsilon} & , a \le t \le a+\varepsilon \\ 0 & , t > a \end{cases}$$

This function is graphically represented in figure below.

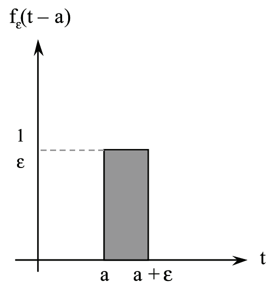


Figure: Unit Impulse Function

In general, Dirac delta function is given by,

$$\delta(t-a) = \underset{\varepsilon \to 0}{Lt}\{f_\varepsilon(t-a)\}$$

Q10. State first shifting theorem.

Answer : Model Paper-2, Q9

Let $f(t)$ be a function defined for all the positive values of 't'. Then, the first shifting theorem states that,

If $L[f(t)] = \bar{f}(s)$, then

$$L[e^{at} f(t)] = \bar{f}(s-a), \text{ where } (s-a) > 0$$

Q11. Find the Laplace transform of $\left[\frac{t}{e^t}\right]$

Answer :

Given function is,

$$f(t) = \frac{t}{e^t}$$

$$\Rightarrow \quad f(t) = t.e^{-t}$$

Applying Laplace transform on both sides,

$$L\{f(t)\} = L\{t.e^{-t}\}$$

$$= \frac{1}{(s+1)^2} \qquad \left[\because L[t^n e^{-at}] = \frac{n!}{(s+a)^{n+1}}\right]$$

$$\therefore \quad L\{t.e^{-t}\} = \frac{1}{(s+1)^2}$$

Q12. Find $L(\sqrt{t}\, e^{-3t})$.

Answer :

Given function is,

$$\sqrt{t}.\, e^{-3t}$$

From the first shifting theorem,

$$L\{t^n.e^{at}\} = \frac{n!}{(s-a)^{n+1}}$$

$$\Rightarrow \quad L\{t^n.e^{at}\} = \frac{\Gamma(n+1)}{(s-a)^{n+1}} \quad [\because \Gamma(n+1) = n!]$$

$$\Rightarrow \quad L\left(t^{\frac{1}{2}}.e^{-3t}\right) = \frac{\Gamma\left(\frac{1}{2}+1\right)}{(s+3)^{\frac{1}{2}+1}}$$

$$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{(s+3)^{\frac{3}{2}}} \quad [\because \Gamma(n+1) = n\Gamma(n)]$$

$$= \frac{\frac{1}{2}.\sqrt{\pi}}{(s+3)^{\frac{3}{2}}} \quad \left[\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right]$$

$$= \frac{\sqrt{\pi}}{2(s+3)^{\frac{3}{2}}}$$

$$\therefore \quad L\left\{\sqrt{t}\, e^{-3t}\right\} = \frac{\sqrt{\pi}}{2(s+3)^{\frac{3}{2}}}$$

Q13. Find the Laplace transform of $t^2 e^{-3t}$

Answer :

Given function is, $t^2 e^{-3t}$

From shifting property,

$$L[e^{at} t^n] = \frac{n!}{(s-a)^{n+1}}$$

Here, $n = 2$ and $a = -3$

$$L[t^2 e^{-3t}] = \frac{2!}{(s+3)^{2+1}}$$

$$= \frac{2}{(s+3)^3}$$

$$\therefore \quad L[t^2 e^{-3t}] = \frac{2}{(2+3)^3}$$

Q14. Find the Laplace transform of e^{-t} cos 2t.

Answer :

Given function is,

$$f(t) = (e^{-t} \cos 2t) \qquad \text{... (1)}$$

$$L\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2 + b^2} \qquad \text{... (2)}$$

Comparing equations (1) and (2), $a = -1, b = 2$

$$\therefore \quad L\{f(t)\} = L(e^{-t} \cos 2t)$$

$$= \frac{s-(-1)}{(s-(-1))^2 + 2^2}$$

$$= \frac{s+1}{(s+1)^2 + 4}$$

$$\therefore \quad L\{e^{-t} \cos 2t\} = \frac{s+1}{(s+1)^2 + 4}$$

Q15. Define second shifting theorem.

Answer :

Statement

Let $f(t)$ be a function defined for all the positive values of 't'.

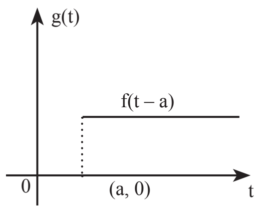
Then, the second shifting theorem states that,

If $L[f(t)] = \bar{f}(s)$ and

$$g(t) = \begin{cases} f(t-a) & ; t > a \\ 0 & ; t < a \end{cases}$$

Then the Laplace transform of $g(t)$ is given by,

$$L[g(t)] = e^{-as} \bar{f}(s)$$



Another Form

If $L\{F(t)\} = \bar{f}(s)$ and $H(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$

then, $L\{F(t-a)\, H(t-a)\} = e^{-as} \bar{f}(s)$

Q16. Find L (f(t)) where f(t) = $\begin{cases} \cos\left(t - \frac{2\pi}{3}\right) & \text{if } t > \frac{2\pi}{3} \\ 0 & \text{if } t < \frac{2\pi}{3} \end{cases}$.

Answer :

Given function is,

$$f(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right), & t > \frac{2\pi}{3} \\ 0 \quad , & t < \frac{2\pi}{3} \end{cases} \quad \ldots (1)$$

Applying Laplace transform,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt = \int_0^{2\pi/3} e^{-st} f(t)dt + \int_{2\pi/3}^\infty e^{-st}(t)dt$$

$$= \int_0^{2\pi/3} e^{-st}(0)dt + \int_{2\pi/3}^\infty e^{-st}\cos\left(t - \frac{2\pi}{3}\right)dt$$

$$= 0 + \int_{2\pi/3}^\infty e^{-st}\cos\left(t - \frac{2\pi}{3}\right)dt$$

$$= \int_{2\pi/3}^\infty e^{-st}\cos\left(t - \frac{2\pi}{3}\right)dt \quad \ldots (2)$$

Let,

$$t - \frac{2\pi}{3} = u$$

$$\Rightarrow \quad t = u + \frac{2\pi}{3}$$

$$\Rightarrow \quad dt = du$$

For Lower Limits

$$\text{For } t = \frac{2\pi}{3}, u = \frac{2\pi}{3} - \frac{2\pi}{3} = 0$$

For Upper Limits

$$t = \infty, u = \infty - \frac{2\pi}{3} = \infty.$$

Substituting the corresponding values in equation (2),

$$L[f(t)] = \int_0^\infty e^{-s\left(u + \frac{2\pi}{3}\right)}\cos u.du = \int_0^\infty e^{-su}.e^{-\frac{2\pi}{3}s}\cos u.du$$

$$= e^{\frac{-2\pi}{3}s}.\int_0^\infty e^{-su}.\cos u.du$$

$$= e^{\frac{-2\pi}{3}s}.L[\cos u] \left(\because \int_0^\infty e^{-su}\cos t\,dt = L[\cos t]\right)$$

$$= e^{-\frac{2\pi}{3}.s}\left(\frac{s}{s^2+1}\right)\left(\because L\{\cos at\} = \frac{s}{s^2+a^2}\right)$$

$$\therefore L[f(t)] = e^{\frac{-2\pi}{3}.s}\left(\frac{s}{s^2+1}\right)$$

Q17. Find L{e^{t-3}u(t – 3)}.

Answer : Model Paper-2, Q10

Given that,

$$e^{t-3}\,u(t-3)$$

Let,

$$f(t) = e^t\,u(t)$$

$$L\{f(t)\} = \int_0^\infty e^t u(t) e^{-st}dt$$

$$F(s) = \int_0^\infty e^{-t(s-1)}\,dt$$

$$= \left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^\infty$$

$$= \frac{e^{-\infty}}{-(s-1)} + \frac{1}{s-1}.e^0$$

$$= \frac{1}{s-1} \qquad [\because e^{-\infty} = 0]$$

$$\therefore \quad F(s) = \frac{1}{s-1}$$

From the property of Laplace transform,

$$L\{f(t-a)\,u(t-a)\} = e^{-as}\,F(s) \quad \ldots (1)$$

Where,

$$L\{f(t)\} = F(s)$$

Here,

$$a = 3$$

Substituting the corresponding values in equation (1),

$$L\left\{f(t-3)u(t-3)\right\} = e^{-3s}.\frac{1}{s-1} = \frac{e^{-3s}}{s-1}$$

$$\therefore L\left\{f(t-3)u(t-3)\right\} = \frac{e^{-3s}}{s-1}$$

Q18. State the Laplace transforms of derivatives and integrals of functions.

Answer :

Laplace Transform of Derivatives of Functions

Statement

If $f(t)$ is continuous function of exponential order and is represented by,

$$L\{f(t)\} = \bar{f}(s)$$

Then,

$$L\{f^n(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0)....- f^{n-1}(0)$$

Laplace Transform of Integrals of Functions

Statement

If $f(t)$ is continuous function and is represented by,

$$L\{f(t)\} = \bar{f}(s)$$

Then,

$$L\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\bar{f}(s)$$

In general,

$$L\left\{\int_0^t\int_0^t......\int_0^t f(u)dudu\right\} = \frac{1}{s^n}\bar{f}(s)$$

Q19. State the Laplace transform of functions when they are multiplied or divided by 't'.

Answer :

Laplace Transform of Function when they are Multiplied by 't'

Statement

If $f(t)$ is a continuous function of exponential order and is represented by,

$$L\{f(t)\} = \bar{f}(s)$$

Then,

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{\bar{f}(s)\}$$

Where, $n = 1, 2, 3....$

Laplace Transform of Functions when they are Divided by 't'

Statement

If $f(t)$ is a continuous function and is represented by,

$$L\{f(t)\} = \bar{f}(s)$$

Then,

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)ds$$

Q20. Find Laplace transform of t sinh t.

Answer :

Given that,

$$f(t) = t\sinh t \quad ...(1)$$

Applying Laplace transform equation (1),

$$L[f(t)] = L[t\sinh t]$$

By multiplication property of Laplace transforms,

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s)$$

$$\Rightarrow \quad L[t\sinh t] = (-1)^1 . \frac{d^1}{ds^1}\left[\frac{1}{s^2-1}\right]$$

$$\left[\because L[f(t)] = L[\sinh t],\ F(s) = \frac{1}{s^2-1}\right]$$

$$= (-1).\frac{d}{ds}\left[\frac{1}{s^2-1}\right]$$

$$= (-1).\left\{\frac{d}{ds}[(s^2-1)^{-1}]\right\}$$

$$= (-1).\left[(-1).(s^2-1)^{-1-1}.\frac{d}{ds}(s^2-1)\right]$$

$$= (-1).[[(-1)(s^2-1)^{-2}.[2s-0]]]$$

$$= (-1)[[(-1)(s^2-1)^{-2}(2s)]]$$

$$= (-1)\left[\frac{(-1)(2s)}{(s^2-1)^2}\right]$$

$$= -1\left[\frac{-2s}{(s^2-1)^2}\right]$$

$$= \frac{2s}{(s^2-1)^2}$$

$$\therefore L[t\sinh t] = \frac{2s}{(s^2-1)^2}$$

Q21. Evaluate $L\{t\,e^{2t}\}$.

Answer :

Given function is, $t\,e^{2t}$

$$L[t\,e^{2t}] = (-1)^1 \frac{d^1}{ds^1}\left[L[e^{2t}]\right]$$

$$\left[\because L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[\overline{f}(s)]\right]$$

$$= (-1)\frac{d}{ds}\left[\frac{1}{s-2}\right] \quad \left[\because L[e^{at}] = \frac{1}{s-a}\right]$$

$$= (-1)\frac{d}{ds}[s-2]^{-1}$$

$$= (-1)(-1)(s-2)^{-1-1}$$

$$= (s-2)^{-2}$$

$$= \frac{1}{(s-2)^2}$$

$$\therefore L[t\,e^{2t}] = \frac{1}{(s-2)^2}$$

Q22. Find $L\left[\frac{\sin t}{t}\right]$

Answer :

Given function is,

$$L\left[\frac{\sin t}{t}\right]$$

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}.ds$$

$$= \left[\tan^{-1}(s)\right]_0^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s)$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$= \cot^{-1} s$$

$$\therefore \quad L\left\{\frac{\sin t}{t}\right\} = \cot^{-1} s$$

Q23. State final value theorem

Answer :

Let $L[f(t)] = F(s)$ and if the Laplace transforms of $f(t)$ and $f'(t)$ exists then $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$.

Q24. If L{F(t)} = $\frac{1}{p(p+\beta)}$ where 'β' is a constant, then find $\lim_{t\to\infty} F(t)$.

Answer :

Given,

$$L\{F(t)\} = \frac{1}{p(p+\beta)}$$

According to final value theorem,

If $L\{F(t)\} = F(p)$, then $\underset{t\to\infty}{Lt} F(t) = \underset{p\to 0}{Lt}\, pF(p)$

$$\Rightarrow \quad \underset{t\to\infty}{Lt} F(t) = \underset{p\to 0}{Lt} (p)\frac{1}{p(p+\beta)}$$

$$= \underset{p\to 0}{Lt} \frac{1}{p+\beta}$$

$$= \frac{1}{\beta}$$

$$\therefore \quad \underset{t\to\infty}{Lt} F(t) = \frac{1}{\beta}$$

Q25. Define inverse Laplace transform.

Answer :

If $\bar{f}(s)$ is considered as the Laplace transform of $f(t)$, then $f(t)$ is called as inverse Laplace transform of $\bar{f}(s)$.

i.e.,

$$f(t) = L^{-1}\{\bar{f}(s)\}$$

Where,

L^{-1} – Inverse Laplace transform operator.

Q26. State inverse Laplace theorem of derivatives and integrals.

Answer :

Inverse Laplace Theorem of Derivatives

Statement

If $f(t)$ is the inverse Laplace transform of $\bar{f}(s)$.

i.e., $L^{-1}\{\bar{f}(s)\} = f(t)$

then,

$$L^{-1}\{\bar{f}^{(n)}s\} = (-1)^n\, t^n f(t)$$

Where,

$$\bar{f}^{(n)}(s) = \frac{d^n}{ds^n}[\bar{f}(s)]$$

Inverse Laplace Theorem of Integrals

Statement

If $f(t)$ is the inverse Laplace transform of $\bar{f}(s)$ then,

$$L^{-1}\left\{\int_s^\infty \bar{f}(s)ds\right\} = \frac{f(t)}{t}$$

Q27. If $L^{-1}\left\{\frac{3}{\left(s-\frac{\pi}{2}\right)^4}\right\}$.

Answer :

Let,

$$F(s) = \frac{3}{\left(s-\frac{\pi}{2}\right)^4}$$

Applying inverse Laplace transform on both sides,

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{3}{\left(s-\frac{\pi}{2}\right)^4}\right\} = 3\,L^{-1}\left\{\frac{1}{\left(s-\frac{\pi}{2}\right)^4}\right\} = 3.L^{-1}\left\{\frac{1}{\left(s-\frac{\pi}{2}\right)^{3+1}}\right\} = 3.\frac{t^3}{3!}.e^{\left(\frac{\pi}{2}\right)t} \qquad \left[\because L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}\right]$$

$$= 3.\frac{t^3}{3\times 2}.e^{\left(\frac{\pi}{2}\right)t} = \frac{t^3}{2}.e^{\left(\frac{\pi}{2}\right)t}$$

$$\therefore\ L^{-1}\left\{\frac{3}{\left(s-\frac{\pi}{2}\right)^4}\right\} = \frac{t^3}{2}.e^{\left(\frac{\pi}{2}\right)t}.$$

Q28. Find inverse Laplace transform of $\frac{s-1}{s^2+5^2}$.

Answer :

$$L^{-1}\left\{\frac{s-1}{s^2+5^2}\right\} = L^{-1}\left[\frac{s}{s^2+5^2}\right] + L^{-1}\left[\frac{-1}{s^2+5^2}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+5^2}\right] - L^{-1}\left[\frac{1}{s^2+5^2}\right]$$

$$= \cos 5t - \frac{1}{5}\sin 5t$$

$$= \frac{5\cos 5t - \sin 5t}{5}$$

$$\therefore\ L^{-1}\left\{\frac{s-1}{s^2+5^2}\right\} = \frac{5\cos 5t - \sin 5t}{5}.$$

Q29. If $L^{-1}\left\{\frac{e^{-\frac{1}{s}}}{s^{\frac{1}{2}}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$ then find $L^{-1}\left\{\frac{e^{-\frac{a}{s}}}{s^{\frac{1}{2}}}\right\}$.

Answer :

Given,

$$L^{-1}\left\{\frac{e^{-1/s}}{s^{1/2}}\right\} = \frac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$$

By change of scale property,

$$L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right) \quad \text{... (1)}$$

$$\Rightarrow \quad L^{-1}\left\{\frac{e^{-1/as}}{(as)^{1/2}}\right\} = \frac{\frac{1}{a}\cos 2\sqrt{\frac{t}{a}}}{\sqrt{\frac{\pi t}{a}}} \quad [\because \text{ From equation (1)}]$$

$$\Rightarrow \quad \frac{1}{\sqrt{a}} L^{-1}\left\{\frac{e^{-1/as}}{s^{1/2}}\right\} = \frac{\sqrt{a}\cos 2\sqrt{\frac{t}{a}}}{a.\sqrt{\pi t}}$$

$$\Rightarrow \quad L^{-1}\left\{\frac{e^{-1/as}}{\sqrt{s}}\right\} = \frac{1}{\sqrt{a}}\frac{\cos 2\sqrt{\frac{t}{a}}}{\sqrt{\pi t}}$$

Let,

$$a' = \frac{1}{a}$$

$$\Rightarrow \quad L^{-1}\left\{\frac{e^{-a'/s}}{\sqrt{s}}\right\} = \frac{\cos(2\sqrt{a't})}{\sqrt{\pi t}}$$

$$\Rightarrow \quad L^{-1}\left\{\frac{e^{-a/s}}{\sqrt{s}}\right\} = \frac{\cos(2\sqrt{at})}{\sqrt{\pi t}}$$

$$\therefore \quad L^{-1}\left\{\frac{e^{-a/s}}{s^{1/2}}\right\} = \frac{\cos(2\sqrt{at})}{\sqrt{\pi t}}$$

Q30. Find $L^{-1}\left\{\frac{1}{s-6} - \frac{2}{s^2+3} + \frac{3}{s^4}\right\}$.

Answer :

$$L^{-1}\left\{\frac{1}{s-6} - \frac{2}{s^2+3} + \frac{3}{s^4}\right\}$$

$$\Rightarrow \quad L^{-1}\left\{\frac{1}{s-6} - \frac{2}{s^2+3} + \frac{3}{s^4}\right\} = L^{-1}\left\{\frac{1}{s-6}\right\} - L^{-1}\left\{\frac{2}{s^2+3}\right\} + L^{-1}\left\{\frac{3}{s^4}\right\}$$

$$= L^{-1}\left\{\frac{1}{s-6}\right\} - 2L^{-1}\left\{\frac{2}{s^2+3}\right\} + 3L^{-1}\left\{\frac{1}{s^4}\right\}$$

$$= e^{6t} - 2.L^{-1}\left[\frac{1}{s^2+(\sqrt{3})^2}\right] + 3.L^{-1}\left[\frac{1}{s^{3+1}}\right]$$

$$= e^{6t} - 2.\frac{1}{\sqrt{3}}\sin\sqrt{3}\,t + \frac{3.t^3}{3!}$$

$$= e^{6t} - \frac{2}{\sqrt{3}}\sin\sqrt{3}\,t + \frac{t^3}{2}$$

$$\therefore \; L^{-1}\left[\frac{1}{s-6} - \frac{2}{s^2+3} + \frac{3}{s^4}\right] = e^{6t} - \frac{2}{\sqrt{3}}\sin\sqrt{3}\,t + \frac{t^3}{2}.$$

Q31. Find $L^{-1}\{\cot^{-1}(s)\}$

Answer : Model Paper-3, Q9

Given that,

$L^{-1}[\cot^{-1}(s)]$

Let,

$$f(t) = L^{-1}[\cot^{-1}(s)]$$

$$\Rightarrow \quad Lf(t) = \cot^{-1}(s)$$

$$\Rightarrow \quad L[t\,f(t)] = (-1)\frac{d}{ds}[\cot^{-1}(s)]$$

$$\Rightarrow \quad L[t\,f(t)] = -\frac{d}{ds}[\cot^{-1}(s)]$$

$$L[t\,f(t)] = \frac{1}{s^2+1} \quad \left[\because \frac{d}{dx}(\cot^{-1}(x)) = \frac{-1}{1+x^2}\right]$$

Applying inverse Laplace transform on both sides,

$$L^{-1}[L[f(t)]] = L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$\Rightarrow \quad tf(t) = L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$\Rightarrow \quad tf(t) = \sin t \quad \left[\because L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}\right]$$

$$\Rightarrow \quad f(t) = \frac{\sin t}{t}$$

$$\therefore \quad L^{-1}[\cot^{-1}(s)] = \frac{\sin t}{t}$$

Q32. Find the inverse Laplace transform of $\frac{e^{-\pi s}}{(s-1)^2}$

Answer :

Given expression is,

$$\frac{e^{-\pi s}}{(s-1)^2}$$

Let,

$$f(t) = \frac{e^{-\pi s}}{(s-1)^2}$$

$$L^{-1}\left[\frac{1}{(s-1)^2}\right] = t.e^t$$

From second shifting theorem,

$$L^{-1}\{e^{-as}.\bar{F}(s)] = F(t-a).U(t-a)$$

$$\Rightarrow L^{-1}\left[\frac{e^{-\pi s}}{(s-1)^2}\right] = (t-\pi).e^{(t-\pi)}.U(t-\pi)$$

$$\therefore \quad L^{-1}\left[\frac{e^{-\pi s}}{(s-1)^2}\right] = (t-\pi).e^{(t-\pi)}.U(t-\pi)$$

Q33. State convolution theorem.

Answer : Model Paper-3, Q10

Statement

Let $f(t)$ and $g(t)$ be two functions for $t > 0$. If $\bar{f}(s)$ and $\bar{g}(s)$ are the Laplace transforms of these functions then,

$$L\{\bar{f}(t)*g(t)\} = \bar{f}(s)\cdot\bar{g}(s)$$

Q34. Apply convolution theorem to evaluate $L^{-1}\left[\frac{1}{s(s+1)}\right]$

Answer :

Given function is, $\frac{1}{s(s+1)}$

Let,

$$\bar{f}(s) = \frac{1}{s} \text{ and } \bar{g}(s) = \frac{1}{s+1}$$

$$\therefore \quad f(t) = L^{-1}\left\{\frac{1}{s}\right\} = 1 \text{ and } g(t) = L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

By convolution theorem,

$$L^{-1}\left[\frac{1}{s(s+1)}\right] = L^{-1}\left\{\frac{1}{s}.\frac{1}{s+1}\right\}$$

$$= L^{-1}\{\bar{f}(s).\bar{g}(s)\}$$

$$= f(t) * g(t)$$

$$= \int_0^t f(u)g(t-u)du$$

$$= \int_0^t 1.e^{-(t-u)}du$$

$$= \int_0^t e^{-t+u}du$$

$$= e^{-t}\left[\int_0^t e^u du\right]$$

$$= e^{-t}[e^u]_0^t$$

$$= e^{-t}[e^t - e^0]$$

$$= e^{-t}[e^t - 1]$$

$$= e^{t-t} - e^{-t}$$

$$= e^0 - e^{-t}$$

$$= 1 - e^{-t}$$

$$\therefore \quad L^{-1}\left[\frac{1}{s(s+1)}\right] = 1 - e^{-t}$$

PART-B

ESSAY QUESTIONS WITH SOLUTIONS

5.1 LAPLACE TRANSFORMS, INVERSE LAPLACE TRANSFORMS, PROPERTIES OF LAPLACE TRANSFORMS AND INVERSE LAPLACE TRANSFORMS

Q35. State the sufficient conditions for the existence of Laplace transform.

Answer :

Existence of Laplace Transform

Laplace transform of any function $f(t)$ exists, if it satisfies the following two conditions.

Condition 1: f(t) is Piece-wise Continuous

A function is said to be piece-wise continuous in an interval $[a, b]$, if it is continuous and has finite limits at the extreme points, in any sub-interval range of $[a, b]$.

Condition 2: f(t) is of Exponential Order 'b'

A function, $f(t)$ is said to be of exponential order 'b', if there exists 'M' and 'b' such that,

$$|f(t)| < Me^{bt}$$

Q36. Show that x^n is of exponential order as $x \to \infty$, n > 0.

Answer :

Given expression is,

$$\underset{x\to\infty}{Lt}\ e^{-ax}x^n = \underset{x\to\infty}{Lt}\ \frac{x^n}{e^{ax}}$$

$$= \frac{\infty}{\infty}$$

Applying L'Hospital rule,

$$\underset{x\to\infty}{Lt}\ e^{-ax}x^n = \underset{x\to\infty}{Lt}\ \frac{nx^{n-1}}{ae^{ax}}$$

$$= \frac{\infty}{\infty}$$

$$= \underset{x\to\infty}{Lt}\ \frac{nx^{n-1}}{ae^{ax}}\left[\frac{\infty}{\infty}\ \text{i.e., Indeterminant form}\right]$$

Repeating the same procedure,

$$\underset{x\to\infty}{Lt}\ e^{-ax}x^n = \underset{x\to\infty}{Lt}\ \frac{n!}{a^n e^{ax}}$$

$\therefore$ x^n is of exponential order.

Q37. State and prove Laplace transforms of elementary functions.

Answer :

Laplace transforms of Elementary Functions

1. $L\{k\} = \dfrac{k}{s} (s > 0)$

Proof

From the definition of Laplace transform,

$$L\{x(t)\} = \int_0^\infty e^{-st}x(t)dt$$

$$\Rightarrow \quad L\{k\} = \int_0^{\infty} ke^{-st}dt$$

$$= k\int_0^{\infty} e^{-st}dt$$

$$= k\left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$= \frac{-k}{s}[e^{-\infty} - e^0]$$

$$= \frac{-k}{s}[0-1] = \frac{k}{s}$$

$$\therefore L\{k\} = \frac{k}{s}$$

2. $L\{t\} = \frac{1}{s^2}$

Proof :

According to the definition of Laplace transform,

$$L[x(t)] = \int_0^{\infty} x(t)e^{-st}dt$$

$$\Rightarrow \quad L\{t\} = \int_0^{\infty} t.e^{-st}dt$$

$$= \left[t\int_0^{\infty} e^{-st}dt - \left[\frac{d}{dt}(1)\int e^{-st}dt\right]dt\right]_0^{\infty}$$

$$= \left[t\frac{e^{-st}}{-s} - \int\left[\frac{e^{-st}}{-s}(1)\right]dt\right]_0^{\infty}$$

$$= \left[t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^{\infty}$$

$$= \left[(\infty)\frac{e^{-\infty}}{-s} - \frac{e^{-\infty}}{s^2}\right] - \left[0.\frac{e^0}{-s} - \frac{e^{-0}}{s^2}\right]$$

$$= [0] + \frac{1}{s^2} = \frac{1}{s^2}$$

$$\therefore \; L\{t\} = \frac{1}{s^2}$$

3. $L\{t^n\} = \frac{n!}{s^{n+1}}$

Proof:

From the definition of Laplace transform,

$$L\{x(t)\} = \int_0^{\infty} x(t)e^{-st}dt$$

$$\Rightarrow \quad L\{t^n\} = \int_0^{\infty} t^n e^{-st}dt$$

$$= \left[t^n\int e^{-st}dt - \int\frac{d}{dt}t^n\int e^{-st}dt\right]dt$$

$$= \left[t^n\frac{e^{-st}}{-s} - \int\left(nt^{n-1}\frac{e^{-st}}{-s}\right)dt\right]_0^{\infty}$$

Applying limits to the first term of above expression becomes zero,

$$\therefore \quad L\{t^n\} = \frac{n}{s}\int_0^{\infty} e^{-st}t^{n-1}dt$$

$$= \frac{n}{s}L\{t^{n-1}\}$$

Similarly,

$$L\{t^{n-1}\} = \frac{n-1}{s}L\{t^{n-2}\}$$

$$L\{t^{n-2}\} = \frac{n-2}{s}L\{t^{n-3}\}$$

By continuously applying Laplace transform, the Laplace transform of n^{th} term is given as,

$$L\{t^n\} = \frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}......\frac{2}{s}.\frac{1}{s}L\{t^{n-n}\}$$

$$= \frac{n!}{s^n}L\{1\}$$

$$= \frac{n!}{s^n}\frac{1}{s} = \frac{n!}{s^{n+1}}$$

$$\therefore \; L\{t^n\} = \frac{n!}{s^{n+1}}$$

4. $L\{e^{at}\} = \frac{1}{s-a}$, $(s - a > 0)$

Proof:

From the definition of Laplace transform,

$$L\{x(t)\} = \int_0^{\infty} x(t)e^{-st}dt$$

$$\Rightarrow \quad L\{e^{at}\} = \int_0^{\infty} e^{-st}e^{at}dt$$

$$\Rightarrow \int_0^{\infty} e^{-(s-a)t} dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty}$$

$$= -\left[\frac{e^{-\infty}}{(s-a)} - \frac{e^0}{(s-a)}\right] = -\left[0 - \frac{1}{s-a}\right] = \frac{1}{s-a}, \text{ if } s > a$$

$$\therefore L\{e^{at}\} = \frac{1}{s-a}$$

5. L{sinh at} = $\frac{a}{s^2 - a^2}$, if s > |a|

Proof:

Given function is,

$f(t) = \sinh at, t \geq 0$

Applying Laplace transform on both sides,

$\therefore L[f(t)] = L[\sinh at]$

$$= L\left[\frac{e^{at} - e^{-at}}{2}\right] \qquad \left[\because \sinh at = \frac{e^{at} - e^{-at}}{2}\right]$$

$$= \frac{1}{2}\left[L[e^{at}] - L[e^{-at}]\right] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a-s+a}{s^2 - a^2}\right]$$

$$= \frac{1}{2}\left[\frac{2a}{s^2 - a^2}\right]$$

$$= \frac{a}{s^2 - a^2}$$

$$\therefore L[\sinh at] = \frac{a}{s^2 - a^2}$$

6. L{cosh at} = $\frac{s}{s^2 - a^2}$, Re (s) > a

Proof:

From the linearity property of the Laplace transform,

$$L\{ax_1(t) + bx_2(t)\} \leftrightarrow aX_1(s) + bX_2(s)$$

$$L\{\cosh at\} = L\left\{\frac{e^{at} + e^{-at}}{2}\right\}$$

$$= \frac{1}{2}\left\{L\{e^{at}\} + L\{e^{-at}\}\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} + \frac{1}{s+a}\right\} \qquad \left[\because L\{e^{at}\} = \frac{1}{s-a}, \; L\{e^{-at}\} = \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left\{\frac{s+a+s-a}{(s-a)(s+a)}\right\} = \frac{1}{2}\left\{\frac{2s}{s^2-a^2}\right\}$$

$$= \frac{s}{s^2-a^2}$$

$$\therefore \quad L\{\cosh at\} = \frac{s}{s^2-a^2}$$

7. L{sin at} = $\mathbf{\frac{a}{s^2+a^2}}$, if s > 0

Proof:

From the definition of Laplace transform,

$$L\{x(t)\} = \int_0^\infty x(t)e^{-st}\,dt$$

$$\Rightarrow \quad L\{\sin at\} = \int_0^\infty \sin at\, e^{-st}\,dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^\infty \qquad \left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= \left[\frac{e^{-\infty}}{s^2+a^2}(-s\sin a(\infty) - a\cos a\infty) - \frac{e^0}{s^2+a^2}[-s\sin a(0) - a\cos a(0)]\right]$$

$$= \left[0 - \frac{1}{s^2+a^2}[-a]\right] = \frac{a}{s^2+a^2}$$

$$\therefore \quad L\{\sin at\} = \frac{a}{s^2+a^2}$$

8. L{cos at} = $\mathbf{\frac{s}{s^2+a^2}}$, if s > 0

Proof:

From the definition of Laplace transform,

$$L\{x(t)\} = \int_0^\infty x(t)e^{-st}\,dt$$

$$\Rightarrow \quad L\{\cos at\} = \int_0^\infty \cos at\, e^{-st}\,dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2}(a\sin at + (-s)\cos at)\right]_0^\infty \qquad \left[\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(b\sin bx + a\cos bx\right]$$

$$= \left[\frac{e^{-s(\infty)}}{s^2+a^2}(a\sin a(\infty) + (-s)\cos a(\infty)\right] - \left[\frac{e^{-s(0)}}{s^2+a^2}(a\sin a(0) + (-s)\cos a(0)\right]$$

$$= 0 - \left[\frac{-s}{s^2+a^2}\right] = \frac{s}{s^2+a^2}$$

$$\therefore \quad L\{\cos at\} = \frac{s}{s^2+a^2}$$

Q38. Find the Laplace transform of f(t) = sinh ωt, t ≥ 0.

Answer :

Given function is,

$f(t)= \sinh \omega t,\ t \geq 0$

Applying Laplace transform on both sides,

$$\therefore\ L[f(t)] = L[\sinh \omega t]$$

$$= L\left[\frac{e^{\omega t} - e^{-\omega t}}{2}\right] \qquad \left[\because \sinh \omega t = \frac{e^{\omega t} - e^{-\omega t}}{2}\right]$$

$$= \frac{1}{2}\left[L[e^{\omega t}] - L[e^{-\omega t}]\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-\omega} - \frac{1}{s+\omega}\right] = \frac{1}{2}\left[\frac{s+\omega-s+\omega}{s^2-\omega^2}\right] = \frac{1}{2}\left[\frac{2\omega}{s^2-\omega^2}\right]$$

$$= \frac{\omega}{s^2-\omega^2}$$

$$\therefore\ L[\sinh \omega t] = \frac{\omega}{s^2-\omega^2}$$

Q39. Find the Laplace transform of f(t) where $f(t) = \begin{cases} t, & 0 < t < \frac{1}{2} \\ t-1, & \frac{1}{2} < t < 1 \\ 0, & t > 1 \end{cases}$

Answer :

Given that,

$$f(t) = \begin{cases} t, & 0 < t < \frac{1}{2} \\ t-1, & \frac{1}{2} < t < 1 \\ 0, & t > 1 \end{cases}$$

Applying Laplace transform,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

$$= \int_0^{\frac{1}{2}} e^{-st} f(t)dt + \int_{\frac{1}{2}}^{1} e^{-st} f(t)dt + \int_1^\infty e^{-st} f(t)dt$$

Substituting $f(t)$ values for different intervals,

$$= \int_0^{\frac{1}{2}} te^{-st} dt + \int_{\frac{1}{2}}^{1} (t-1)e^{-st} dt + \int_1^\infty e^{-st}(0)dt$$

$$= \int_0^{\frac{1}{2}} te^{-st}\,dt + \int_{\frac{1}{2}}^{1} (t-1)e^{-st}\,dt + 0$$

$$= \int_0^{\frac{1}{2}} te^{-st}\,dt + \int_{\frac{1}{2}}^{1} te^{-st}\,dt - \int_{\frac{1}{2}}^{1} e^{-st}\,dt$$

$$= \left(t.\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right)_0^{\frac{1}{2}} + \left(t.\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right)_{\frac{1}{2}}^{1} - \left(\frac{e^{-st}}{-s}\right)_{\frac{1}{2}}^{1}$$

$$= \left[\left(\frac{1}{2}.\frac{e^{-\frac{1}{2}s}}{-s} - \frac{e^{-\frac{1}{2}s}}{s^2}\right) - \left(0 - \frac{e^0}{s^2}\right)\right] + \left[\left(\frac{1.e^{-s}}{-s} - \frac{e^{-s}}{s^2}\right) - \left(\frac{1}{2}\frac{e^{-\frac{1}{2}s}}{-s} - \frac{e^{-\frac{1}{2}s}}{s^2}\right)\right] - \left[\frac{e^{-s}}{-s} - \frac{e^{-\frac{1}{2}s}}{-s}\right]$$

$$= -\frac{1}{2}.\frac{e^{-\frac{1}{2}s}}{-s} - \frac{e^{-\frac{1}{2}s}}{s^2} + \frac{1}{s^2} + \frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2} - \frac{1}{2}\frac{e^{-\frac{1}{2}s}}{-s} + \frac{e^{-\frac{1}{2}s}}{s^2} - \frac{e^{-s}}{-s} + \frac{e^{-\frac{1}{2}s}}{-s}$$

$$= -\frac{1}{2}\frac{e^{\frac{-1}{2}s}}{s} - \frac{e^{\frac{-1}{2}s}}{s^2} + \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{2}\frac{e^{\frac{-1}{2}s}}{s} + \frac{e^{\frac{-1}{2}s}}{s^2} + \frac{e^{-s}}{s} - \frac{e^{\frac{-1}{2}s}}{s}$$

$$= \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-\frac{1}{2}s}}{s}$$

$$= \frac{1}{s^2}(1 - e^{-s}) - \frac{e^{-\frac{1}{2}s}}{s}$$

$$\therefore\ L\,[f(t)] = \frac{1}{s^2}(1 - e^{-s}) - \frac{e^{-\frac{1}{2}s}}{s}$$

Q40. Find L[3 cos 3t cos 4t].

Answer :

Given function is, $L[3 \cos 3t \cos 4t]$

Consider, $\cos 3t \cos 4t$

Multiplying and dividing by 2,

$$\cos 3t.\cos 4t = \frac{1}{2}(2\cos 3t.\cos 4t)$$

$$= \frac{1}{2}[\cos(3t + 4t) + \cos(3t - 4t)] \qquad [\because 2\cos A\cos B = \cos(A+B) + \cos(A-B)]$$

$$= \frac{1}{2}[\cos 7t + \cos t\,(-t)]$$

$$= \frac{1}{2}[\cos 7t + \cos t]$$

$$= \frac{1}{2}\cos 7t + \frac{1}{2}\cos t$$

$$\Rightarrow \quad \cos 3t \cos 4t = \frac{1}{2}\cos 7t + \frac{1}{2}\cos t \qquad \text{... (1)}$$

Applying Laplace transform on both sides,

$$L\{3\cos 3t\cos 4t\} = L\{3(\cos 3t.\cos 4t)\}$$

$$= L\left\{3\left(\frac{1}{2}\cos 7t + \frac{1}{2}\cos t\right)\right\}$$

[∵ From equation (1)]

$$= L\left\{\frac{3}{2}\cos 7t + \frac{3}{2}\cos t\right\}$$

$$= L\left\{\frac{3}{2}\cos 7t\right\} + L\left\{\frac{3}{2}\cos t\right\}$$

$$= \frac{3}{2}L\{\cos 7t\} + \frac{3}{2}L\{\cos t\}$$

$$= \frac{3}{2}\left(\frac{s}{s^2+7^2}\right) + \frac{3}{2}\left(\frac{s}{s^2+1}\right)$$

$$\left[\because L\{\cos at\} = \frac{s}{s^2+a^2}\right]$$

$$= \frac{3}{2}\left[\frac{s}{s^2+49} + \frac{s}{s^2+1}\right]$$

$$\therefore\ L\{3\cos 3t\ \cos 4t\} = \frac{3}{2}\left[\frac{s}{s^2+49} + \frac{s}{s^2+1}\right]$$

Q41. Find the Laplace Transform of the following functions,

(i) f(t) = k^t, k is a real constant >1

(ii) f(t) = tδ'(t).

Answer :

(i) f(t) = k^t, k is a Real Constant >1

Given that,

$f(t) = k^t$

Where,

K is real constant > 1

By the definition of Laplace transform,

$$\bar{f}(s) = L[f(t)] = \int_0^\infty k^t\, e^{-st}\, dt \quad \text{... (1)}$$

$$\Rightarrow \quad \bar{f}(s) = \left[k^t \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} k^t.\log k\, dt$$

$$\left[\because \frac{d}{dx}(a^x) = a^x \log a\right]$$

$$\Rightarrow \quad \bar{f}(s) = \frac{1}{s} + \frac{\log k}{s}\int_0^\infty e^{-st}\, k^t\, dt$$

$$\Rightarrow \quad \bar{f}(s) = \frac{1}{s} + \frac{\log k}{s}.\bar{f}(s)$$

[∵ From equation (1)]

$$\Rightarrow \quad \bar{f}(s)\left[1 - \frac{\log k}{s}\right] = \frac{1}{s}$$

$$\Rightarrow \quad \bar{f}(s)\left[\frac{s-\log k}{s}\right] = \frac{1}{s}$$

$$\Rightarrow \quad \bar{f}(s) = \frac{1}{(s-\log k)}$$

$$\therefore L[k^t] = \frac{1}{(s-\log k)}$$

(ii) f(t) = tδ'(t)

Given that,

$f(t) = t\,\delta'(t)$

$$\int t.\delta'(t)dt = t.\int \delta'(t).dt - \int \left(t' \int \delta'(t).dt\right) dt$$

$$= t.\delta(t) - \int 1.\delta(t).dt \quad \text{... (2)}$$

Here, $\delta(t) = 1$ for $t = 0$. But, at the same instant $t.\delta(t) = 0$.

Substituting above value in equation (2),

$$\int t.\delta'(t).dt = t.(0) - \int \delta(t).dt$$

$$\Rightarrow \quad \int t.\delta'(t).dt = 0 - \int \delta(t).dt$$

$$\Rightarrow \quad \int t.\delta'(t).dt = -\int \delta(t).dt$$

$$\therefore \quad t.\delta'(t) = -\delta(t) \quad \text{... (3)}$$

By definition of Laplace transform,

$$F(s) = L[f(t)]$$

$$= L[t.\delta'(t)]$$

$$= L[-\delta(t)] \quad [\because \text{From equation (3)}]$$

$$\Rightarrow \quad F(s) = \int_0^\infty -\delta(t).e^{-st}dt$$

And also, for a unit impulse function or signal,

$$\delta(t) = \begin{cases} 1, & t = 0 \\ 0, & \text{else where} \end{cases}$$

$$\Rightarrow \quad \bar{f}(s) = -(1) - e^{-st}\big|_{t=0} = -e^{-s(0)} = -1$$

$$\therefore \quad L[t.\delta'(t)] = L[-\delta(t)] = -1$$

Q42. Define unit step function and find the Laplace transform of unit step function.

Answer :

Unit Step Function

Unit step function is defined as $U(t-a)$.

If $t < a$, $U(t-a) = 0$

If $t \geq a$, $U(t-a) = 1$

Where, $a \geq 0$

Laplace Transform of Unit Step Function L{u(t – a)}

By definition of Laplace transform,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$\therefore \quad L[U(t-a)] = \int_0^\infty e^{-st} U(t-a) dt$$

$$= \int_0^a e^{-st} U(t-a)\, dt + \int_a^\infty e^{-st} U(t-a)\, dt \quad ...(1)$$

$$= \int_0^a e^{-st}(0)\, dt + \int_a^\infty e^{-st}(1)\, dt$$

$$= \int_a^\infty e^{-st}\, dt = \frac{[e^{-st}]_a^\infty}{-s} = \frac{e^{-as}}{s}$$

$$\therefore L[U(t-a)] = \frac{e^{-as}}{s}$$

Note

If $a = 0$, $L[U(t)] = \dfrac{1}{s}$

Q43. Define Dirac's delta function or the unit impulse function. Also obtain its Laplace transform.

Answer :

The unit impulse function or Dirac's delta function is the limiting form of,

$$\delta(t-a) = \begin{cases} 0, & t < a \\ \dfrac{1}{\varepsilon}, & a \leq t \leq a+\varepsilon \\ 0, & t > a \end{cases}$$

Integrating,

$$\int_0^\infty \delta(t-a) = \int_0^a \delta(t-a).dt + \int_a^{a+\varepsilon} \delta(t-a).dt + \int_{a+\varepsilon}^\infty \delta(t-a).dt$$

$$= \int_0^a 0.dt + \int_a^{a+\varepsilon} \delta(t-a).dt + \int_{a+\varepsilon}^\infty 0.dt$$

$$= 0 + \int_a^{a+\varepsilon} \delta(t-a).dt + 0$$

$$= \int_a^{a+\varepsilon} \delta(t-a).dt$$

$$\Rightarrow \quad \int_0^\infty \delta t(t-a) = \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\, dt = \frac{1}{\varepsilon}[t]_a^{a+\varepsilon}$$

Hence, the Dirac's delta function or the unit impulse function is defined as,

$$\delta(t-a) = \infty,\ t = a$$
$$= 0,\ t \neq a$$

So that $\int_0^\infty \delta(t-a)dt = 1 \qquad [\because a \geq 0]$

Laplace Transform of Dirac's Delta Function

Let $f(t)$ be continuous at $n = a$.

Then $\int_0^\infty f(t)\, \delta\, \varepsilon\, (t-a)\, dt = \int_\varepsilon^{a+\varepsilon} f(t).\frac{1}{\varepsilon}\, dt$

$= (a + \varepsilon - a).\ f(t).$

$\dfrac{1}{\varepsilon}$ (Where, $a <<< a + \varepsilon$)

As $\varepsilon \to 0$, $\int_0^\infty f(t)\, \delta(t-a) dt = f(a)$

$$L(\delta(t-a)) = \int_0^\infty e^{-st}\, \delta(t-a)\, dt = e^{-sa}$$

$$L(\delta(t)) = e^{-s} = e^0 = 1$$

Q44. State and prove linearity property of Laplace transform.

Answer :

(i) Linearity

$L[x_1(t)] \leftrightarrow X_1(s)$

$L[x_2(t)] \leftrightarrow X_2(s)$

$\therefore \quad L[ax_1(t) + bx_2(t)] \leftrightarrow aX_1(s) + bX_2(s)$

Proof

According to the definition of Laplace transform,

$$L[x_1(t)] = \int_{-\infty}^\infty x_1(t)e^{-st} dt = X_1(s)$$

$$L[x_2(t)] = \int_{-\infty}^\infty x_2(t)e^{-st} dt = X_2(s)$$

$$L[ax_1(t)+bx_2(t)] = \int_{-\infty}^{\infty}[ax_1(t)+bx_2(t)]e^{-st}dt$$

$$= \int_{-\infty}^{\infty} ax_1(t)e^{-st}dt + \int_{-\infty}^{\infty} bx_2(t)e^{-st}dt$$

$$= a\int_{-\infty}^{\infty} x_1(t)e^{-st}dt + b\int_{-\infty}^{\infty} x_2(t)e^{-st}dt$$

$$= aX_1(s) + bX_2(s)$$

$$\therefore \quad L[ax_1(t)+bx_2(t)] = aX_1(s)+bX_2(s)$$

Q45. If L[f(t)] = F(s) then prove that $L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$

Answer :

Theorem

If $L\{f(t)\} = \bar{f}(s)$ = then,

$$L\{f(at) = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$$

Proof: Since, $L\{f(at) = \int_0^{\infty} e^{-st}f(at)dt$

Let,

$at = u$, $adt = du$ i.e., $dt = \frac{du}{a}$

Limits are $t = 0$

$\Rightarrow u = 0$; $t = \infty \Rightarrow u = \infty$

$$L\{f(at)\} = \frac{1}{a}\int_0^{\infty} e^{-\left(\frac{s}{a}\right)u} f(u)\,du$$

$$\Rightarrow \quad L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$$

$$\therefore \quad L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$$

Q46. State and prove first shifting theorem.

Answer :

Statement

For answer refer Unit-5, Q10.

Proof

Given that,

$$L\{f(t)\} = \bar{f}(s) \quad \text{... (1)}$$

Since,

$$L\{f(t)\} = \int_0^{\infty} e^{-st}f(t)dt$$

$$\therefore \quad L\{e^{at}f(t)\} = \int_0^{\infty} e^{-st}[e^{at}f(t)]dt$$

$$= \int_0^{\infty} e^{-(s-a)t}f(t)\,dt \quad \text{... (2)}$$

Comparing equations (1) and (2),

$$L\{e^{at}f(t)\} = \bar{f}(s-a)$$

Q47. Prove that,

(i) $L[e^{at}\sinh bt] = \frac{b}{(s-a)^2-b^2}$

(ii) $L[e^{at}\cosh bt] = \frac{s-a}{(s-a)^2-b^2}$.

Answer :

(i) $L[e^{at}\sinh bt] = \frac{b}{(s-a)^2-b^2}$

Consider, $L[\sinh bt]$

Since, $L[f(t)] = \int_0^{\infty} e^{-st}.f(t)dt$

$$\Rightarrow \quad L[\sinh bt] = \int_0^{\infty} e^{-st}.\sinh bt\,dt$$

$$= \int_0^{\infty} e^{-st}\left(\frac{e^{bt}-e^{-bt}}{2}\right)dt$$

$$\left(\because \sinh(at) = \frac{e^{at}-e^{-at}}{2}\right)$$

$$= \frac{1}{2}\int_0^{\infty}(e^{-st}.e^{bt}-e^{-st}e^{-bt})dt$$

$$= \frac{1}{2}\left[\int_0^{\infty} e^{-(s-b)t}dt - \int_0^{\infty} e^{-(s+b)t}dt\right]$$

$$= \frac{1}{2}\left[\left(\frac{e^{-(s-b)t}}{-(s-b)}\right)_0^{\infty} - \left(\frac{e^{-(s+b)t}}{-(s+b)}\right)_0^{\infty}\right]$$

$$= \frac{1}{2}\left[\left(\frac{e^{-\infty}-e^0}{-(s-b)}\right) - \left(\frac{e^{-\infty}-e^0}{-(s+b)}\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{0-1}{-(s-b)}\right) - \left(\frac{0-1}{-(s+b)}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-b} - \frac{1}{s+b}\right] = \frac{1}{2}\left[\frac{s+b-(s-b)}{(s+b)(s-b)}\right]$$

$$= \frac{1}{2}\left[\frac{2b}{s^2-b^2}\right]$$

$\Rightarrow \quad L[\sinh(bt)] = \dfrac{b}{s^2 - b^2}$

$L[e^{at} \sinh bt] = \dfrac{b}{(s-a)^2 - b^2}$ $[\because \; L[e^{at} f(t)] = \bar{f}(s-a)]$

$\therefore \quad L[e^{at} \sinh bt] = \dfrac{b}{(s-a)^2 - b^2}$

(ii) $\mathbf{L(e^{at} \cosh bt) = \dfrac{s-a}{(s-a)^2 - b^2}}$

Consider, $L[\cosh bt]$

Since, $L[f(t)] = \int_0^\infty e^{-st} f(t) dt$

$$\Rightarrow L[\cosh bt] = \int_0^\infty e^{-st} \cosh bt \, dt$$

$$= \int_0^\infty e^{-st} \left(\frac{e^{bt} + e^{-bt}}{2} \right) dt \qquad \left(\because \cosh at = \frac{e^{at} + e^{-at}}{2} \right)$$

$$= \frac{1}{2} \int_0^\infty (e^{-st}.e^{bt} + e^{-st}.e^{-bt}) dt$$

$$= \frac{1}{2} \left[\int_0^\infty e^{-(s-b)t} dt + \int_0^\infty e^{-(s+b)t} dt \right]$$

$$= \frac{1}{2} \left[\left(\frac{e^{-(s-b)t}}{-(s-b)} \right)_0^\infty + \left(\frac{e^{-(s+b)t}}{-(s+b)} \right)_0^\infty \right]$$

$$= \frac{1}{2} \left[\frac{e^{-\infty} - e^0}{-(s-b)} + \frac{e^{-\infty} - e^0}{-(s+b)} \right]$$

$$= \frac{1}{2} \left[\frac{1}{s-b} + \frac{1}{s+b} \right]$$

$$= \frac{1}{2} \left[\frac{s+b+s-b}{(s-b)(s+b)} \right]$$

$$= \frac{1}{2} \left(\frac{2s}{s^2 - b^2} \right)$$

$\Rightarrow \quad L[\cosh bt] = \dfrac{s}{s^2 - b^2}$

$\Rightarrow \quad L[e^{at} \cosh bt] = \dfrac{(s-a)}{(s-a)^2 - b^2}$ $[\because \; L[e^{at} f(t)] = \bar{f}(s-a)]$

$\therefore \quad L[e^{at} \cosh bt] = \dfrac{(s-a)}{(s-a)^2 - b^2}$

Q48. Evaluate $L\left\{e^t\left(\cos 2t + \frac{1}{2}\sinh 2t\right)\right\}$.

Answer :

Given function is,

$$L\left\{e^t\left(\cos 2t + \frac{1}{2}\sinh\ 2t\right)\right\}$$

Applying Laplce transform on both sides,

$$L\left\{e^t\cos 2t + \frac{1}{2}e^t\sinh\ 2t\right\} = L\{e^t\cos 2t\} + \frac{1}{2}L\{e^t\sinh\ 2t\}$$

$$= \frac{(s-1)}{(s-1)^2+2^2} + \frac{1}{2}\left(\frac{2}{(s-1)^2-2^2}\right) \quad \left[\because L[e^{at}\cos bt] = \frac{(s-a)}{(s-a)^2+b^2},\ L[e^{at}\sinh bt] = \frac{b}{(s-a)^2-b^2}\right]$$

$$= \frac{s-1}{s^2-2s+1+4} + \frac{1}{s^2-2s+1-4}$$

$$= \frac{s-1}{s^2-2s+5} + \frac{1}{s^2-2s-3}$$

$$\therefore\ L\left\{e^t\left(\cos 2t + \frac{1}{2}\sinh\ 2t\right)\right\} = \frac{s-1}{s^2-2s+5} + \frac{1}{s^2-2s-3}$$

Q49. Find the Laplace transform of the function e^{-3t} (2 cos 5t – 3 sin 5t).

Answer :

Given function is,

$$e^{-3t}(2\cos 5t - 3\sin 5t)$$

The Laplace transform of above expression can be obtained as,

$$L[e^{-3t}(2\cos 5t - 3\sin 5t] = L[e^{-3t}(2\cos 5t)] - L[e^{-3t}.(3\sin 5t)]$$

$$= 2L[e^{-3t}.\cos 5t] - 3\,L[e^{-3t}.\sin 5t]$$

$$= 2\left[\frac{s+3}{(s+3)^2+25}\right] - 3\left[\frac{5}{(s+3)^2+25}\right] \quad \left(\because\ L(e^{-at}\cos bt) = \frac{s+a}{(s+a)^2+b^2},\ L(e^{-at}\sin bt) = \frac{b}{(s+a)^2+b^2}\right)$$

$$= \frac{2(s+3)}{(s+3)^2+25} - \frac{15}{(s+3)^2+25}$$

$$= \frac{2s+6-15}{(s+3)^2+25}$$

$$= \frac{2s-9}{(s+3)^2+25}$$

$$\therefore\ L[e^{-3t}(2\cos 5t - 3\sin 5t)] = \frac{2s-9}{(s+3)^2+25}$$

Q50. State and prove second shifting theorem.

Answer :

Statement

For answer refer Unit-5, Q15.

Proof: Given function is,

$$L[f(t)] = \bar{f}(s) \quad \text{... (1)}$$

And, $g(t) = \begin{cases} f(t-a) & ; t > a \\ 0 & ; t < a \end{cases}$ ... (2)

According to the definition of Laplace transform,

$$L[f(t)] = \int_0^\infty e^{-st}.f(t)dt \quad \text{... (3)}$$

$$\therefore \quad L[g(t)] = \int_0^\infty e^{-st}.g(t)dt$$

$$= \int_0^a e^{-st}\ .g(t)dt + \int_a^\infty e^{-st}.g(t)dt$$

$$= \int_0^a e^{-st}.0dt + \int_a^\infty e^{-st}.f(t-a)dt \qquad [\because \text{From equation (2)}]$$

$$= 0 + \int_a^\infty e^{-st} f(t-a)dt$$

$$\therefore \quad L\{g(t)] = \int_a^\infty e^{-st} f(t-a)dt \quad \text{... (4)}$$

Let, $t - a = x \Rightarrow t = x + a$

$dt - 0 = dx$

$\Rightarrow \quad dt = dx$

Lower Limit: When $t = a$; $x = 0$

Upper Limit: When $t = \infty$; $x = \infty$

Substituting the corresponding values in equation (4),

$$L[g(t)] = \int_0^\infty e^{-s(x+a)} f(x)dx$$

$$= \int_0^\infty e^{-sx}.e^{-sa} f(x)dx$$

$$= e^{-sa}\int_0^\infty e^{-sx} f(x)dx$$

$$= e^{-sa}L[f(t)] \qquad [\because \text{From equation (3)}]$$

$$= e^{-sa}.\bar{f}(s) \qquad [\because \text{From equation (1)}]$$

$$\therefore \ L[g(t)] = e^{-sa}\overline{f}(s)$$

Q51. Show that $L\{t^n(f(t))\} = (-1)^n \frac{d^n}{ds^n}\{\bar{f}(s)\}$ where, n = 1, 2, 3,.....

OR

Prove that $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{\bar{f}(s)\}$ where, n = 1, 2, 3,.....

Answer :

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\{\bar{f}(s)\} \quad \text{... (1)}$$

where, $n = 1, 2, 3,.....$

By the definition of Laplace transform,

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= L[f(t)]$$

Differentiating on both sides with respect to s,

$$\frac{d}{ds}\bar{f}(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt$$

$$= \int_0^\infty \frac{d}{ds} e^{-st} f(t)dt$$

$$= \int_0^\infty - te^{-st} f(t)dt = -\int_0^\infty e^{-st}[t f(t)]dt$$

$$= -L[t f(t)]$$

$$\Rightarrow \quad L[t f(t)] = -\frac{d}{ds}\bar{f}(s) \quad \text{... (2)}$$

By mathematical induction, equation (2) can be written as,

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\bar{f}(s)$$

Q52. Find L[t sin 3t cos 2t].

Answer : Model Paper-1, Q15(a)

Consider, $t \sin 3t \cos 2t$

$$\Rightarrow \quad t \sin 3t \cos 2t = \frac{t}{2}[2 \sin 3t \cos 2t] \quad [\because \text{ Multiplying and dividing by 2}]$$

$$= \frac{t}{2}[\sin(3t + 2t) + \sin(3t - 2t)] \quad [\because \sin(A + B) + \sin(A - B) = 2 \sin A \cos B]$$

$$= \frac{t}{2}[\sin 5t + \sin t]$$

$$\Rightarrow \quad t \sin 3t \cos 2t = \frac{1}{2}[t \sin 5t + t \sin t]$$

Applying Laplace transform on both sides,

$$L[t \sin 3t \cos 2t] = L\left[\frac{1}{2}[t \sin 5t + t \sin t\right]$$

$$= \frac{1}{2}[L(t \sin 5t) + L(t \sin t)]$$

$$= \frac{1}{2}L\{t\sin 5t\} + \frac{1}{2}L\{t\sin t\}$$

$$= \frac{1}{2}\left[(-1)\frac{d}{ds}\left(\frac{5}{s^2+5^2}\right)\right] + \frac{1}{2}(-1)\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{1}{2}\ (-1)(5)\frac{d}{ds}\left(\frac{1}{s^2+5^2}\right) + \frac{(-1)}{2}\left[\frac{(s^2+1)(0)-2s}{(s^2+1)^2}\right]$$

$$= \frac{-5}{2}\left[\frac{(s^2+5^2)(0)-(1)(2s)}{(s^2+5^2)^2}\right] + \frac{(-1)(-2s)}{2(s^2+1)^2}$$

$$= \frac{-5}{2}\left[\frac{-2s}{(s^2+5^2)^2}\right] + \frac{s}{(s^2+1)^2}$$

$$= \frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}$$

$$\therefore\ L[t\sin 3t\cos 2t] = \frac{5s}{(s^2+25^2)} + \frac{s}{(s^2+1)^2}$$

Q53. Find the Laplace transform of te^{-t} sin 2t.

Answer :

Given expression is,

$te^{-t}\sin 2t$

$$L\{te^{-t}\sin 2t\} = (-1)\frac{d}{ds}L\{e^{-t}\sin 2t\} \qquad \left[\because L\{t^n f(t)\} = (-1)^n\frac{d^n}{ds^n}[f(s)]\right]$$

$$= (-1)\frac{d}{ds}\frac{2}{(s+1)^2+2^2} \qquad \left[\because L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}\right]$$

$$= (-1)\frac{d}{ds}\left[\frac{2}{s^2+1+2s+4}\right]$$

$$= (-1)\frac{d}{ds}\left[\frac{2}{s^2+2s+5}\right]$$

$$= (-1)\ \frac{0-2(2s+2)}{(s^2+2s+5)^2}$$

$$= \frac{2(2s+2)}{(s^2+2s+5)^2}$$

$$= \frac{4(s+1)}{(s^2+2s+5)^2}$$

$$\therefore L\{te^{-t}\sin 2t\} = \frac{4(s+1)}{(s^2+2s+5)^2}$$

Q54. Find the Laplace transform of [t cost sinh2t].

Answer :

Given expression is,

$t.\cos t.\sinh 2t$

$$\Rightarrow \quad L\{t.\cos t.\sinh 2t] = -\frac{d}{ds}L\{\cos t.\sinh 2t\} \qquad \left[\because L\{t.F(t)\} = -\frac{d}{ds}F(s)\right] \qquad \dots (1)$$

Consider,

$L\{\cos t.\sinh 2t\}$

$$\Rightarrow \quad L\{\cos t.\sinh 2t\} = L\left\{\cos t\left(\frac{e^{2t}-e^{-2t}}{2}\right)\right\}$$

$$= L\left\{\frac{\cos t.e^{2t}}{2} - \frac{e^{-2t}\cos 2t}{2}\right\}$$

$$= \frac{1}{2}L\{\cos t.e^{2t} - e^{-2t}\cos 2t\}$$

$$= \frac{1}{2}[L\{e^{2t}\cos 2t\} - L\{e^{-2t}\cos 2t\}]$$

$$= \frac{1}{2}\left[\frac{s-2}{(s-2)^2+1^2} - \frac{(s+2)}{(s+2)^2+1^2}\right]$$

$$= \frac{1}{2}\left[\frac{s-2}{(s-2)^2+1^2} - \frac{(s+2)}{(s+2)^2+1^2}\right]$$

Substituting the corresponding values in equation (1),

$$L\{t.\cos t.\sinh 2t\} = -\frac{1}{2}\frac{d}{ds}\left[\frac{(s-2)}{(s-2)^2+1^2} - \frac{(s+2)}{(s+2)^2+1^2}\right]$$

$$= -\frac{1}{2}\left[\frac{d}{ds}\left(\frac{(s-2)}{(s-2)^2+1^2}\right) - \frac{d}{ds}\left(\frac{(s+2)}{(s+2)^2+1^2}\right)\right]$$

$$= -\frac{1}{2}\left[\left[\frac{((s-2)^2+1^2)(1)-(s-2)2(s-2)}{(s-2)^2+1^2}\right] - \frac{(s+2)^2+1^2(1)-(s+2)2(s+2)(1)}{(s+2)^2+1^2}\right]$$

$$\therefore \quad L\{t.\cos t.\sinh 2t\} = -\frac{1}{2}\left[\frac{1-(s-2)^2}{(s-2)^2+1^2} - \frac{1-(s+2)^2}{(s+2)^2+1^2}\right]$$

Q55. If L[f(t)] = $\bar{f}$(s), then prove that $L\left[\frac{[f(t)]}{t}\right] = \int_s^\infty \bar{f}(s)ds$ provided $\lim_{t\to 0}\frac{f(t)}{t}$ exists.

Answer :

Given that,

$$L\{f(t)\} = \bar{f}(s)$$

And $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)ds$ for $\underset{t\to 0}{Lt}\frac{f(t)}{t}$ exists

By the definition of Laplace transform,

$$L\{f(t)\} = \bar{f}(s) = \int_0^\infty e^{-st}f(t)dt$$

Integrating on both sides with respect to 's',

$$\therefore \quad \int_s^\infty \bar{f}(s)ds = \int_s^\infty \left[\int_0^\infty e^{-st} f(t)dt\right]ds$$

$$= \int_0^\infty \int_s^\infty e^{-st} f(t)\, ds\; dt \qquad [\because \text{ Changing the order of integration}]$$

$$= \int_0^\infty f(t)\left[\int_s^\infty e^{-st} ds\right]dt \qquad [\because f(t) \text{ is independent of } s]$$

$$= \int_0^\infty f(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt$$

$$= \int_0^\infty f(t)\left[-\frac{1}{t}(e^{-\infty} - e^{-st})\right]dt$$

$$= \int_0^\infty f(t)\left[-\frac{1}{t}(0 - e^{-st})\right]dt$$

$$= \int_0^\infty f(t)\left[\frac{e^{-st}}{t}\right]dt$$

$$= \int_0^\infty e^{-st} \frac{f(t)}{t} dt = L\left\{\frac{f(t)}{t}\right\}$$

$$\therefore \quad L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)ds$$

Q56. Prove that $L\left\{\int_0^t f(u)\, du\right\} = \frac{1}{s}\bar{f}(s)$ where $L\{f(t)\} = \bar{f}(s)$.

Answer :

Given that,

$$L[f(t)] = \bar{f}(s)$$

Consider,

$$L\int_0^t f(u)\, du = \int_0^\infty e^{-st}\left[\int_0^t f(u)\, du\right]dt$$

$$= \left[\frac{e^{-st}}{-s}\int_0^t f(u)\, du\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} f(t)\, dt$$

$$= 0 + \frac{1}{s} L[f(t)] \qquad \left(\because \int_0^0 f(u)\, du = 0\right)$$

$$= \frac{1}{s}\bar{f}(s)$$

$$\therefore \quad L\int_0^t f(u)\, du = \frac{\bar{f}(s)}{s}$$

Q57. Find the Laplace transform of $\frac{\cos at - \cos bt}{t}$.

Answer : Model Paper-2, Q15(a)

Given function is,

$$f(t) = \frac{\cos at - \cos bt}{t} \quad \text{... (1)}$$

Since,

$$L[\cos at] = \frac{s}{s^2+a^2}, \quad L[\cos bt] = \frac{s}{s^2+b^2}$$

$$L[\cos at - \cos bt] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

The Laplace transform of equation (1) can be written as,

$$L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right).ds \qquad \left(\because L\left[\frac{1}{t}.f(t)\right] = \int_s^\infty F(s).ds\right)$$

$$= \int_s^\infty \frac{s}{s^2+a^2}.ds - \int_s^\infty \frac{s}{s^2+b^2}.ds \quad \text{... (2)}$$

$$= \frac{1}{2}\left[\int_s^\infty \frac{2s}{s^2+a^2}ds - \int_s^\infty \frac{2s}{s^2+b^2}ds\right]$$

$$= \frac{1}{2}[\log(s^2+a^2) - \log(s^2+b^2)]_s^\infty \qquad \left[\because \int \frac{f'(x)}{f(x)}dx = \log f(x)\right]$$

$$= \frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{1+\frac{a^2}{\infty}}{1+\frac{b^2}{\infty}} - \log\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right]$$

$$= \frac{1}{2}\left[\log\frac{1+0}{1+0} - \log\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right] = \frac{1}{2}\left[\log 1 - \log\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right]$$

$$= \frac{1}{2}\left[0 - \log\frac{s^2+a^2}{s^2+b^2}\right]$$

$$= -\frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2} = \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

$$\therefore \quad L\left[\frac{\cos at - \cos bt}{t}\right] = \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

Q58. Using Laplace transform, evaluate $\int_0^\infty \frac{e^{-at}\sin^2 t}{t}dt$.

Answer : Model Paper-3, Q15(b)

Given integral is,

$$\int_0^\infty e^{-at}.\frac{\sin^2 t}{t}dt$$

The above integral is exactly the Laplace of $\frac{\sin^2 t}{t}$ with $s = a$.

$$L\left\{\frac{\sin^2 t}{t}\right\}\Bigg|_{s=a} = \int_0^\infty \frac{e^{-at}.\sin^2 t}{t}dt\Bigg|_{s=a} \quad \text{... (1)}$$

Consider $\sin^2 t$

$$\Rightarrow \quad L[\sin^2 t] = L\left\{\frac{1-\cos 2t}{2}\right\} \qquad \left(\because \sin^2\theta = \frac{1-\cos 2\theta}{2}\right)$$

$$= \frac{1}{2}L\{1\} - \frac{1}{2}L\{\cos 2t\}$$

$$= \frac{1}{2}.\frac{1}{s} - \frac{1}{2}.\frac{s}{s^2+4} \qquad \left(\because L\{1\} = \frac{1}{s},\ L\{\cos 2t\} = \frac{s}{s^2+a^2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$\Rightarrow \quad L\left\{\frac{\sin^2 t}{t}\right\} = \int_s^\infty \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)ds \qquad \left(\because L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)ds\right)$$

$$\Rightarrow \quad L\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{2}\int_s^\infty \left[\frac{1}{s}ds - \frac{s}{s^2+4}ds\right]$$

$$= \frac{1}{2}\int_s^\infty \left[\frac{1}{s}ds - \frac{2s}{2(s^2+4)}ds\right] = \frac{1}{2}\left[\int_s^\infty \frac{1}{s}ds - \frac{1}{2}\int_s^\infty \frac{2s}{s^2+4}ds\right]$$

$$= \left[\frac{1}{2}\log(s) - \frac{1}{4}\log(s^2+4)\right]_s^\infty \qquad \left(\because \int \frac{2s}{s^2+a^2}ds = \log(s^2+a^2)\right)$$

$$= \left[\log(s)^{1/2} - \log(s^2+4)^{1/4}\right]_s^\infty$$

$$= \log\left[\frac{\sqrt{s}}{(s^2+4)^{1/4}}\right]_s^\infty = -\log\left[\frac{\sqrt{s}}{(s^2+4)^{1/4}}\right]$$

$$= \log\left[\frac{(s^2+4)^{1/4}}{(s^2)^{1/4}}\right] = \frac{1}{4}\log\left[\frac{s^2+4}{s^2}\right]$$

$$\Rightarrow \quad L\left\{\frac{\sin^2 t}{t}\right\} = \int_0^\infty e^{-at}.\frac{\sin^2 t}{t}dt = \frac{1}{4}\log\left[\frac{s^2+4}{s^2}\right]$$

Substituting $s = a$ in above equation,

$$L\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{4}\log\left\{\frac{a^2+4}{a^2}\right\}$$

$$\therefore \int_0^\infty e^{-at}\frac{\sin^2 t}{t}dt = \frac{1}{4}\log\left\{\frac{a^2+4}{a^2}\right\}$$

Q59. Find L$\left[\frac{e^{-t}\sin t}{t}\right]$.

Answer :

Given expression is,

$$\left[\frac{e^{-t}\sin t}{t}\right]$$

Consider, sin t

$$L[\sin t] = \frac{1}{s^2+1} \qquad \left(\because L[\sin at] = \frac{a}{s^2+a^2}\right)$$

$$L[e^{-t}\sin t] = \frac{1}{(s+1)^2+1} \qquad \left(\because L[e^{at}\sin bt] = \frac{b}{(s-a)^2+b^2}\right)$$

$$L\left[\frac{e^{-t}\sin t}{t}\right] = \int_s^\infty \frac{ds}{(s+1)^2+1^2} \qquad \left(\because \text{If } L[f(t)] = \bar{f}(s), \text{ then } L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)ds\right)$$

$$= \left[\tan^{-1}\left(\frac{s+1}{1}\right)\right]_s^\infty \qquad \left[\because \int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s+1)$$

$$= \pi/2 - \tan^{-1}(s+1) \qquad [\because \tan^{-1}\infty = \pi/2]$$

$$= \cot^{-1}(s+1) \qquad \left(\because \cot^{-1}(\theta) = \frac{\pi}{2} - \tan^{-1}\theta\right)$$

$$\therefore L\left[\frac{e^{-t}\sin t}{t}dt\right] = \cot^{-1}(s+1)$$

Q60. Find L$\left[\frac{\cos 4t \sin 2t}{t}\right]$.

Answer :

Given function is,

$$\left[\frac{\cos 4t\sin 2t}{t}\right]$$

$$\Rightarrow \quad \cos 4t\sin 2t = \frac{\sin(4t+2t)-\sin(4t-2t)}{2} \qquad [\because 2\cos A\sin B = \sin(A+B) - \sin(A-B)]$$

$$= \frac{1}{2}[\sin 6t - \sin 2t]$$

Applying Laplace transform on both sides,

$$L[\cos 4t \sin 2t] = L\left[\frac{\sin 6t - \sin 2t}{2}\right]$$

$$= \frac{1}{2}[L(\sin 6t) - L(\sin 2t)]$$

$$= \frac{1}{2}\left[\frac{6}{s^2+36} - \frac{2}{s^2+4}\right] \qquad \left[\because L[\sin at] = \frac{a}{s^2+a^2}\right]$$

$$= \frac{3}{s^2+36} - \frac{1}{s^2+4}$$

If, $L[f(t)] = \bar{f}(s)$, then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)ds$

$$\Rightarrow \quad L\left[\frac{\cos 4t \sin 2t}{t}\right] = \left[\int_s^\infty \frac{3}{s^2+36} - \int_s^\infty \frac{1}{s^2+4}\right]ds$$

$$= \int_s^\infty \left(\frac{3}{s^2+36}\right)ds - \int_s^\infty \left(\frac{1}{s^2+4}\right)ds$$

$$= \left[\frac{3}{6}\left[\tan^{-1}\left(\frac{s}{6}\right)\right]\right]_s^\infty - \left[\frac{1}{2}\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty \qquad \left[\because \int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right]$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{\infty}{6}\right) - \tan^{-1}\left(\frac{s}{6}\right)\right] - \frac{1}{2}\left[\tan^{-1}\left(\frac{\infty}{2}\right) - \tan^{-1}\left(\frac{s}{2}\right)\right]$$

$$= \frac{1}{2}\left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{6}\right)\right] - \frac{1}{2}\left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{2}\right)\right]$$

$$= \frac{1}{2}\left[\tan^{-1}(\infty) - \tan^{-1}\left(\frac{s}{6}\right) - \tan^{-1}(\infty) + \tan^{-1}\left(\frac{s}{2}\right)\right]$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{s}{2}\right) - \tan^{-1}\left(\frac{s}{6}\right)\right]$$

$$\therefore \quad L\left[\frac{\cos 4t \sin 2t}{t}\right] = \frac{1}{2}\left[\tan^{-1}\left(\frac{s}{2}\right) - \tan^{-1}\left(\frac{s}{6}\right)\right]$$

Q61. If L[f(t)] = F(s), then $\underset{t\to 0}{\text{Lt}}\ (t) = \underset{s\to\infty}{\text{Lt}}\ sF(s)$

Answer :

Given Laplace transform is,

$L[f(t)] = F(s)$

Laplace transform of derivative function is,

$L\{f^n(t)\} = s^nF(s) - s^{n-1} f(0)$

For $n = 1$,

$L\{f'(t)\} = sL\{f(t)\} - s^{1-1}f(0)$

$\Rightarrow \quad L[f'(t)] = sL[f(t)] - f(0)$

$= sF(s) - f(0)$

$\Rightarrow \quad sF(s) - f(0) = L[f'(t)]$

$$= \int_0^\infty e^{-st} f'(t)\,dt$$

Applying limits on both sides,

$$\underset{s\to\infty}{Lt}[sF(s) - f(0)] = \underset{s\to\infty}{Lt}\int_0^\infty e^{-st} f'(t)\,dt$$

$$\Rightarrow \quad \underset{s\to\infty}{Lt}[sF(s) - f(0)] = 0 \qquad [\because e^{-\infty} = 0]$$

$$\Rightarrow \quad \underset{s\to\infty}{Lt}\; sF(s) = f(0) = \underset{s\to\infty}{Lt}\; f(t)$$

$$\therefore \quad \underset{t\to 0}{Lt}\; f(t) = \underset{s\to\infty}{Lt}\; sF(s)$$

Q62. If Lf(t) = F(s), then $\underset{t\to\infty}{\mathbf{Lt}}\; \mathbf{f(t)} = \underset{s\to 0}{\mathbf{Lt}}\; \mathbf{sF(s)}$

Answer : Model Paper-3, Q15(a)

Given Laplace transform is,

$L(f(t)) = F(s)$

Laplace transform of derivative function is,

$$L\{f^n(t)\} = s^nF(s) - s^{n-1}f(0) - ......$$

For $n = 1$,

$$L\{f'(t)\} = sF(s) - s^0 f(0)$$

$$\Rightarrow \quad L\{f'(t)\} = sL[f(t)] - f(0)$$

$$\Rightarrow \quad sL[f(t)] - f(0) = L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$$

Applying limits on both sides

$$\underset{s\to 0}{Lt}[sL[f(t)] - f(0)] = \underset{s\to 0}{Lt}\int_0^\infty e^{-st} f'(t)\,dt$$

$$= \int_0^\infty f'(t)\,dt$$

$$= \int_0^\infty d[f(t)]$$

$$= [f(t)]_0^\infty$$

$$\Rightarrow \quad \underset{s\to 0}{Lt}[sF(s) - f(0)] = f(\infty) - f(0)$$

$$\Rightarrow \quad \underset{s\to 0}{Lt}\; sF(s) = f(\infty) = \underset{t\to\infty}{Lt}\; f(t)$$

$$\therefore \quad \underset{t\to\infty}{Lt}\; f(t) = \underset{s\to 0}{Lt}\; sF(s)$$

Q63. Verify initial and final value theorems for the function f(t) = 1 + e^{-t}(sint + cost)

Answer :

Given function is,

$$f(t) = 1 + e^{-t}(\sin t + \cos t)$$

Applying Laplace transform on both sides,

$$L[f(t)] = L[1 + e^{-t}(\sin t + \cos t)]$$

$$= L[1] + L[e^{-t}\sin t] + L[e^{-t}\cos t]$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$$

$$\left[\because L(e^{-at}\sin bt) = \frac{b}{(s+a)^2+b^2} L[e^{-at}\cos bt] = \frac{s+a}{(s+a)^2+b^2}\right]$$

$$= \frac{1}{s} + \frac{1+s+1}{(s+1)^2+1}$$

$$\Rightarrow \quad F(s) = \frac{1}{s} + \frac{s+2}{(s+1)^2+1}$$

Initial Value Theorem

The initial value theorem is given by,

$$\underset{t\to 0}{Lt}\; f(t) = \underset{s\to\infty}{Lt}\; s.F(s)$$

Consider,

$$\underset{t\to 0}{Lt}\; f(t) = \underset{t\to 0}{Lt}\; (1 + e^{-t}(\sin t + \cos t))$$

$$= 1 + e^0(\sin 0 + \cos 0)$$

$$= 1 + 1(0+1)$$

$$= 1 + 1$$

$$= 2$$

$$\therefore \quad \underset{t\to 0}{Lt}\; f(t) = 2$$

Consider,

$$\underset{s\to\infty}{Lt}\; sF(s) = \underset{s\to\infty}{Lt}\; s\left[\frac{1}{s} + \frac{s+2}{(s+1)^2+1}\right]$$

$$= \underset{s\to\infty}{Lt}\; s\left[\frac{(s+1)^2+1+s(s+2)}{s[(s+1)^2+1]}\right]$$

$$= \underset{s\to\infty}{Lt}\left[\frac{(s+1)^2+s^2+2s+1}{(s+1)^2+1}\right]$$

$$= \underset{s\to\infty}{Lt}\left[\frac{s^2+1+2s+s^2+2s+1}{(s+1)^2+1}\right]$$

$$= \underset{s\to\infty}{Lt}\left[\frac{2s^2+4s+2}{(s+1)^2+1}\right] = \underset{s\to\infty}{Lt}\left[\frac{2s^2+4s+2}{(s^2+2s+2)}\right]$$

$$= \underset{s\to\infty}{Lt}\; \frac{s^2\left[2+\frac{4}{s}+\frac{2}{s^2}\right]}{s^2\left[1+\frac{2}{s}+\frac{2}{s^2}\right]}$$

$$= \underset{s\to\infty}{Lt} \frac{2+\frac{4}{s}+\frac{2}{5}}{1+\frac{2}{s}+\frac{2}{s^2}}$$

$$= \underset{s\to\infty}{Lt} \frac{2\left[1+\frac{2}{s}+\frac{2}{s^2}\right]}{\left[1+\frac{2}{s}+\frac{2}{s^2}\right]}$$

$$= 2$$

$$\Rightarrow \underset{s\to\infty}{Lt}\ sF(s) = 2$$

$$\Rightarrow \underset{t\to 0}{Lt}\ f(t) = \underset{s\to\infty}{Lt}\ sF(s)$$

$\therefore$ Initial value theorem is verified.

Final Value Theorem

The final value theorem is given by,

$$\underset{t\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ sF(s)$$

Consider,

$$\underset{t\to\infty}{Lt}\ f(t) = \underset{t\to\infty}{Lt}\ [1+e^{-t}(\sin t+\cos t)]$$

$$= 1 + \underset{t\to\infty}{Lt}\ e^{-t}(\sin t+\cos t)$$

$$= 1 + e^{-\infty} \qquad [\because e^{-\infty} = 0]$$

$$= 1$$

$$\therefore \underset{t\to\infty}{Lt}\ f(t) = 1$$

Consider,

$$\underset{s\to 0}{Lt}\ sF(s) = \underset{s\to 0}{Lt}\ s\left[\frac{1}{s}+\frac{s+2}{(s+1)^2+1}\right]$$

$$= \underset{s\to 0}{Lt}\left[1+\frac{s(s+2)}{(s+1)^2+1}\right]$$

$$= 1+0$$

$$= 1$$

$$\therefore \underset{s\to 0}{Lt}\ sF(s) = 1$$

$$\Rightarrow \underset{t\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ sF(s)$$

$\therefore$ Final value theorem is verified.

Q64. Verify the initial and final value theorems for f(t) = 3e^{-2t}

Answer :

Given function is,

$$f(t) = 3e^{-2t}$$

Applying Laplace transform on both sides,

$$L\{f(t)\} = 3L\{e^{-2t}\}$$

$$f(s) = 3.\frac{1}{s+2} = \frac{3}{s+2}$$

Initial Value Theorem

The initial value theorem is given as,

$$\underset{t\to 0}{Lt}\ f(t) = \underset{s\to\infty}{Lt}\ s.F(s)$$

Consider,

$$\underset{t\to 0}{Lt}\ f(t) = \underset{t\to 0}{Lt}\ 3e^{-2t}$$

$$= 3.e^{-2(0)} = 3(1) = 3$$

$$\therefore \underset{t\to 0}{Lt}\ f(t) = 3$$

Consider,

$$\underset{s\to\infty}{Lt}\ s.F(s) = \underset{s\to\infty}{Lt}\ s\left[\frac{3}{s+2}\right] = \underset{s\to\infty}{Lt}\left[\frac{3s}{s+2}\right]$$

$$= \underset{s\to\infty}{Lt}\frac{3s}{s\left(1+\frac{2}{s}\right)}$$

$$= \underset{s\to\infty}{Lt}\frac{3}{1+\frac{2}{s}} = \frac{3}{1+0}$$

$$\Rightarrow \underset{s\to\infty}{Lt}\ s.F(s) = 3$$

Here, $\underset{t\to 0}{Lt}\ f(t) = \underset{s\to\infty}{Lt}\ s.F(s) = 3$

$\therefore$ Initial value theorem is verified.

Final Value Theorem

The final value theorem is given as,

$$\underset{t\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ s.F(s)$$

Consider,

$$\underset{t\to\infty}{Lt}\ f(t) = \underset{t\to\infty}{Lt}\ 3e^{-2t} = 3.e^{-\infty}$$

$$= 3(0) \qquad [\because e^{-\infty} = 0]$$

$$\therefore \underset{t\to\infty}{Lt}\ f(t) = 0$$

Consider,

$$\underset{s\to 0}{Lt}\ s.F(s) = \underset{s\to 0}{Lt}\ s.\left[\frac{3}{s+2}\right] = \underset{s\to 0}{Lt}\left[\frac{3s}{s+2}\right]$$

$$\Rightarrow \underset{s\to 0}{Lt}\ s.F(s) = 0$$

Here, $\underset{t\to\infty}{Lt}\ f(t) = \underset{s\to 0}{Lt}\ s.F(s) = 0$

$\therefore$ Final value theorem is verified.

Q65. Define inverse Laplace transform and list some standard inverse Laplace transforms.

Answer :

Inverse Laplace Transform

For answer refer Unit-5, Q25.

List of Some Standard Inverse Laplace Transforms

	$\overline{f}(s)$	$L^{-1}[\overline{f}(s)] = f(t)$
1.	$\frac{1}{s}$	1
2.	$\frac{1}{s^{n+1}}$, n is positive integer	$\frac{t^n}{n!}$
3.	$\frac{1}{s^{n+1}}$, $n > -1$	$\frac{t^n}{\Gamma(n+1)}$
4.	$\frac{1}{s-a}$	e^{at}
5.	$\frac{1}{s+a}$	e^{-at}
6.	$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$
7.	$\frac{s}{s^2+a^2}$	$\cos at$
8.	$\frac{1}{s^2-a^2}$	$\frac{1}{a}\sinh at$
9.	$\frac{s}{s^2-a^2}$	$\cosh at$
10.	$\frac{1}{(s-a)^2+b^2}$ or $\frac{1}{(s+a)^2+b^2}$	$\frac{1}{b}e^{at}\sin bt$ or $\frac{1}{b}e^{-at}\sin bt$
11.	$\frac{s-a}{(s-a)^2+b^2}$ or $\frac{s+a}{(s+a)^2+b^2}$	$e^{at}\cos bt$ or $e^{-at}\cos bt$
12.	$\frac{1}{(s-a)^2-b^2}$ or $\frac{1}{(s+a)^2-b^2}$	$\frac{1}{b}e^{at}\sinh bt$ or $\frac{1}{b}e^{-at}\sinh bt$
13.	$\frac{s-a}{(s-a)^2-b^2}$ or $\frac{s+a}{(s+a)^2-b^2}$	$e^{at}\cosh bt$ or $\frac{1}{b}e^{-at}\cosh bt$
14.	$\frac{2as}{(s^2+a^2)^2}$	$t\sin at$
15.	$\frac{s^2-a^2}{(s^2+a^2)^2}$	$t\cos at$

Q66. Find the $L^{-1}\left[\frac{5s+1}{(s+2)(s-1)}\right]$.

Answer : Model Paper-1, Q15(b)

Given function is,

$$\frac{5s+1}{(s+2)(s-1)}$$

Applying partial fractions,

$$\frac{5s+1}{(s+2)(s-1)} = \frac{A}{s+2}+\frac{B}{s-1} \quad \text{... (1)}$$

$$\Rightarrow \quad 5s+1 = A(s-1)+B(s+2)$$

$$\Rightarrow \quad 5s+1 = As-A+Bs+2B$$

$$\Rightarrow \quad 5s+1 = s(A+B)-A+2B$$

Comparing the coefficients on both sides,

$$A+B=5 \quad \text{... (2)}$$

$$-A+2B=1 \quad \text{... (3)}$$

Solving equations (2) and (3),

$B=2, A=3$

Substituting the values of A and B in equation (1),

$$\frac{5s+1}{(s+2)(s-1)} = \frac{3}{s+2}+\frac{2}{s-1}$$

Applying inverse Laplace transform on both sides,

$$\frac{5s+1}{(s+2)(s-1)} = L^{-1}\left[\frac{3}{s+2}\right]+L^{-1}\left[\frac{2}{s-1}\right]$$

$$= 3e^{-2t}+2e^{t}$$

$$\therefore \quad L^{-1}\left[\frac{5s+1}{(s+2)(s-1)}\right] = 3e^{-2t}+2e^{t}$$

Q67. Find the inverse Laplace transform of the following $\log\left(\frac{s+1}{s-1}\right)$.

Answer :

Given Laplace transform is,

$$F(s) = \log\left(\frac{s+1}{s-1}\right)$$

Applying inverse Laplace transform on both sides of above equation,

$$L^{-1}[F(s)] = L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left[\log\left(\frac{s+1}{s-1}\right)\right]\right] \quad \left[\because L^{-1}[F(s)] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}[\log(s+1)]-\frac{d}{ds}[\log(s-1)]\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{1}{s+1}-\frac{1}{s-1}\right]$$

$$= -\frac{1}{t}[e^{-t} - e^{t}] \qquad \left[\because L^{-1}\left[\frac{1}{s+1}\right] = e^{-t},\ L^{-1}\left[\frac{1}{s-1}\right] = e^{t}\right]$$

$$= \frac{1}{t}[e^{t} - e^{-t}] = \frac{2}{t}\left[\frac{e^{t} - e^{-t}}{2}\right]$$

$$= \frac{2}{t}[\sin h t] \qquad \left[\because \sinh t = \frac{e^{t} - e^{-t}}{2}\right]$$

$$= \frac{2\sin h t}{t}$$

$$\therefore L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right] = \frac{2\sin h\, t}{t}$$

Q68. If $L\{f(t)\} = \frac{20-4s}{s^2-4s+20}$, find (i) $L\{e^{-t}f(2t)\}$ (ii) $L\{f(3t)\}$.

Answer :

Given,

$$L\{f(t)\} = \frac{20-4s}{s^2-4s+20}$$

$$\Rightarrow \quad f(t) = L^{-1}\left[\frac{20-4s}{s^2-4s+20}\right]$$

$$= L^{-1}\left[\frac{20-4s}{s^2-4s+4+16}\right]$$

$$= L^{-1}\left[\frac{20-4s}{(s-2)^2+4^2}\right]$$

$$= L^{-1}\left[\frac{20}{(s-2)^2+4^2}\right] - 4L^{-1}\left[\frac{s}{(s-2)^2+4^2}\right]$$

$$= 5.L^{-1}\left[\frac{4}{(s-2)^2+4^2}\right] - 4\,L^{-1}\left[\frac{(s-2)+2}{(s-2)^2+4^2}\right]$$

$$= 5.e^{2t}\sin 4t - 4\,L^{-1}\left[\frac{s-2}{(s-2)^2+4^2}\right] - 4L^{-1}\left[\frac{2}{(s-2)^2+4^2}\right]$$

$$= 5.e^{2t}\sin 4t - 4.e^{2t}.\cos 4t - 2.L^{-1}\left[\frac{4}{(s-2)^2+4^2}\right]$$

$$= 5.e^{2t}\sin 4t - 4e^{2t}.\cos 4t - 2.e^{2t}\sin 4t$$

$$= 3.e^{2t}\sin 4t - 4\,e^{2t}.\cos 4t$$

$$\Rightarrow \quad f(t) = e^{2t}(3\sin 4t - 4.\cos 4t) \qquad \text{... (1)}$$

$$\therefore \quad f(2t) = e^{4t}(3\sin 8t - 4\cos 8t)$$

(i) L{e^{-t} f(2t)}

$$\Rightarrow \quad L\{e^{-t}[e^{4t}(3\sin 8t - 4\cos 8t)]\}$$

$$\Rightarrow \quad L\{e^{-t}.e^{4t}(3\sin 8t - 4\cos 8t)\}$$

$$\Rightarrow \quad L\{e^{3t}.(3\sin 8t - 4\cos 8t)\}$$

$$= L\{e^{3t}.(3\sin 8t)\} - L\{e^{3t}(4\cos 8t)\}$$

$$= 3L\{e^{3t}\sin 8t\} - 4L\{e^{3t}\cos 8t\}$$

$$= 3.\frac{8}{(s-3)^2+8^2} - 4.\frac{(s-3)}{(s-3)^2+8^2}$$

$$= \frac{24}{s^2-6s+9+64} - \frac{4s-12}{s^2-6s+9+64}$$

$$= \frac{24-4s+12}{s^2-6s+73}$$

$$= \frac{36-4s}{s^2-6s+73}$$

$$\therefore\ L\{e^{-2t}f(2t)\} = \frac{36-4s}{s^2-6s+73}$$

(ii) L{f(3t)}

From equation (1),

$$f(3t) = e^{6t}(3\sin 12t - 4\cos 12t)$$

$$= 3e^{6t}\sin 12t - 4e^{6t}\cos 12t$$

Applying Laplace transform on both sides,

$$L\{f(3t)\} = L[3e^{6t}\sin 12t - 4e^{6t}\cos 12t]$$

$$= 3L[e^{6t}\sin 12t] - 4\,L[e^{6t}\cos 12t]$$

$$= 3\left[\frac{12}{(s-6)^2+(12)^2}\right] - 4\left[\frac{s-6}{(s-6)^2+(12)^2}\right]$$

$$= \frac{36}{(s-6)^2+144} - \frac{4(s-6)}{(s-6)^2+144}$$

$$= \frac{36-4s+24}{(s-6)^2+144} = \frac{60-4s}{(s-6)^2+144}$$

$$\therefore\ L\{f(3t)\} = \frac{60-4s}{(s-6)^2+144}$$

5.2 CONVOLUTION THEOREM (WITHOUT PROOF)

Q69. Find the inverse Laplace Transform of $\frac{s}{(s^2+a^2)^2}$ by using Convolution theorem.

Answer :

Given function is,

$$\frac{s}{(s^2+a^2)^2}$$

Let, $\overline{f}(s) = \frac{s}{s^2+a^2}$

Then, $f(t) = L^{-1}\left[\frac{s}{(s^2+a^2)}\right]$ $\left[\because L^{-1}\left[\frac{s}{(s^2+a^2)}\right] = \cos at\right]$

$= \cos at$

Let, $\overline{g}(s) = \frac{1}{s^2+a^2}$

Then, $g(t) = L^{-1}\left[\frac{1}{(s^2+a^2)}\right]$

$= \frac{1}{a}\sin at$ $\left[\because L^{-1}\left[\frac{1}{(s^2+a^2)}\right] = \frac{1}{a}\sin at\right]$

From convolution theorem,

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \int_0^t f(u)g(t-u)\,du \qquad \left[\because L^{-1}[\overline{f}(s)\overline{g}(s)] = f(t)*g(t) = \int_0^t f(u)g(t-u)\,du\right]$$

$$= (\cos at) * \frac{1}{a}(\sin at)$$

$$= \frac{1}{a}\int_0^t \cos au \sin a(t-u)\,du$$

$$= \frac{1}{a}\int_0^t \frac{[\sin(au+a(t-u)) - \sin(au-a(t-u)]\,du}{2} \qquad \left[\because \cos A \sin B = \frac{\sin(A+B)-\sin(A-B)}{2}\right]$$

$$= \frac{1}{2a}\int_0^t [\sin(au+at-au) - \sin(au-at+au)]\,du$$

$$= \frac{1}{2a}\int_0^t [\sin at - \sin(2au-at)]\,du$$

$$= \frac{1}{2a}\left[\int_0^t \sin at\,du - \int_0^t \sin(2au-at)\,du\right]$$

$$= \frac{1}{2a}\left\{\sin at\big[(u)\big]_0^t + \left[\frac{\cos(2au-at)}{2a}\right]_0^t\right\}$$

$$= \frac{1}{2a}\left\{\big[t\sin at - 0\big] + \left[\frac{\cos(2at-at)}{2a} - \frac{\cos(0-at)}{2a}\right]\right\}$$

$$= \frac{1}{2a}\left[t\sin at + \frac{\cos at}{2a} - \frac{\cos at}{2a}\right]$$

$$= \frac{1}{2a}[t\sin at + 0]$$

$$= \frac{1}{2a}t\sin at$$

$$\therefore\ L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a}t\sin at$$

Q70. Using convolution theorem, find $L^{-1}\left\{\frac{s}{(s^2+4)(s^2+9)}\right\}$

Answer : **Model Paper-2, Q15(b)**

Given that,

$$L^{-1}\left\{\frac{s}{(s^2+4)(s^2+9)}\right\}$$

Let,

$$\overline{f}(s) = \frac{s}{s^2+4} \text{ and } \overline{g}(s) = \frac{1}{s^2+9}$$

$$f(t) = L^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t \qquad \left[\because L^{-1}\left[\frac{s}{s^2+a^2} = \cos at\right]\right]$$

$$g(t) = L^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{\sin 3t}{3} \qquad \left[\because L^{-1}\left[\frac{s}{s^2+a^2}\right] = \frac{1}{a}\sin at\right]$$

By convolution theorem,

$$L^{-1}\left\{\frac{s}{(s^2+4)(s^2+9)}\right\} = L^{-1}\left\{\frac{s}{s^2+4}.\frac{s}{s^2+9}\right\}$$

$$= L^{-1}\{\overline{f}(s).\overline{g}(s)\}$$

$$= f(t) * g(t)$$

$$= (\cos 2t) * \left(\frac{\sin 3t}{3}\right)$$

$$= \frac{1}{3}\int_0^t \cos 2u \sin 3(t-u)du \qquad \boxed{\because \int f(t)*g(t) = \int f(u)g(t-u)du}$$

$$= \frac{1}{3}.\frac{1}{2}\int_0^t 2\cos 2u \sin 3(t-u)du$$

$$= \frac{1}{6}\int_0^t [\sin(-u+3t) - \sin(5u-3t)]du$$

$$= \frac{1}{6}\int_0^t \sin(-u+3t)du - \int_0^t \sin(5u-3t)du$$

$$= \frac{1}{6}\left[-\cos(-u+3t)(-1) - \frac{(-\cos(5u-3t))}{5}\right]_0^t$$

$$= \frac{1}{6\times 5}[5\cos(-u+3t) + \cos(5u-3t)]_0^t$$

$$= \frac{1}{30}[5\times\cos(-t+3t) + \cos(5t-3t)] - [5(\cos(0+3t)) + \cos(0-3t)]]$$

$$= \frac{1}{30}[5\cos(2t) + \cos(2t) - 5\cos 3t - \cos(-3t)]$$

$$= \frac{1}{30}[5 \cos 2t + \cos 2t - 5\cos 3t - \cos 3t]$$

$$= \frac{1}{30}[6 \cos 2t - 6 \cos 3t]$$

$$= \frac{6}{30}[\cos 2t - \cos 3t]$$

$$= \frac{\cos 2t - \cos 3t}{5}$$

$$\therefore L^{-1}\left\{\frac{s}{(s^2+4)(s^2+9)}\right\} = \frac{1}{5}[\cos 2t - \cos 3t]$$

Q71. Using convolution theorem, find $L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\}$

Answer :

Given expression is,

$$\left\{\frac{1}{(s+a)(s+b)}\right\}$$

Let, $\bar{f}(s) = \frac{1}{s+a}$

Then, $f(t) = e^{-at}$

Let, $\bar{g}(s) = \frac{1}{s+b}$

Then, $g(t) = e^{-bt}$

$\Rightarrow \quad \bar{f}(s).\bar{g}(s) = \frac{1}{(s+a)(s+b)}$

Applying inverse Laplace transform on both sides,

$$L^{-1}\{\bar{f}(s).\bar{g}(s)\} = L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\}$$

$$= \int_0^t f(u)\, g(t-u)\, du$$

$$= \int_0^t e^{-au} e^{-b(t-u)} du$$

$$= \int_0^t e^{-au} e^{-bt}.e^{bu} du$$

$$= e^{-bt} \int_0^t e^{(b-a)u} du$$

$$= e^{-bt}\left[\frac{e^{(b-a)u}}{b-a}\right]_0^t$$

$$= \frac{e^{-bt}}{b-a}[e^{(b-a)t} - e^{(b-a)0}]$$

$$= \frac{e^{-bt}}{b-a}[e^{(b-a)t} - 1]$$

$$\therefore \quad L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\} = \frac{e^{-bt}}{b-a}[e^{(b-a)t} - 1].$$

Q72. Using convolution theorem, find the inverse Laplace transform of $\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$

Answer :

Given expression is,

$$\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$$

And,

$$L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at \text{ and } L^{-1}\left[\frac{s}{s^2+b^2}\right] = \cos bt$$

Let,

f(t) = cos at, g(t) = cos bt

By convolution theorem,

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right]$$

$$= L^{-1}\{\bar{f}(s).\bar{g}(s)\}$$

$$= f(t) * g(t)$$

$$= \cos at * \cos bt$$

$$= \int_0^t f(u)g(t-u)\,du \qquad \left[\because \int f(t)*g(t) = \int f(u)g(t-u)du\right]$$

$$= \int_0^t \cos au \cos b(t-u)\,du$$

$$= \frac{1}{2}\int_0^t 2\times\cos au \cos b(t-u)\,du$$

$$= \frac{1}{2}\int_0^t [\cos(au+bt-bu)+\cos(au-bt+bu)]\,du \qquad [\because 2\cos a\cos b = \cos(a+b)+\cos(a-b)]$$

$$= \frac{1}{2}\int_0^t \cos(u(a-b)+bt)+\cos(u(a+b)-bt)\,du$$

$$= \frac{1}{2}\left[\int_0^t \cos(u(a-b)+bt)\,du + \int_0^t \cos(u(a+b)-bt)\,du\right]$$

$$= \frac{1}{2}\left[\frac{\sin(u(a-b)+bt)}{(a-b)}+\frac{\sin(u(a+b)-bt)}{(a+b)}\right]_0^t$$

$$= \frac{1}{2}\left[\left[\frac{\sin(t(a-b)+bt}{(a-b)}+\frac{\sin(t(a+b)-bt}{(a+b)}\right]-\left[\frac{\sin 0(a-b)+bt}{(a-b)}+\frac{\sin(0(a+b)-bt}{(a+b)}\right]\right]$$

$$= \frac{1}{2}\left[\frac{\sin(at-bt+bt)}{(a-b)}+\frac{\sin(at+bt-bt)}{a+b}-\frac{\sin bt}{(a-b)}-\frac{\sin(-bt)}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at}{(a-b)}+\frac{\sin at}{(a+b)}-\frac{\sin bt}{(a-b)}+\frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{(a+b)\sin at+(a-b)\sin at}{(a-b)(a+b)}-\frac{[\sin bt(a+b)-\sin bt(a-b)]}{(a-b)(a+b)}\right]$$

$$= \frac{1}{2}\left[\frac{a\sin at+b\sin at+a\sin at-b\sin at-a\sin bt-b\sin bt+a\sin bt-b\sin bt}{a^2-b^2}\right]$$

$$= \frac{1}{2}\left[\frac{2a\sin at-2b\sin bt}{a^2-b^2}\right]$$

$$= \frac{1}{2}\times 2\left[\frac{a\sin at-b\sin bt}{a^2-b^2}\right]$$

$$= \frac{a\sin at-b\sin bt}{a^2-b^2}$$

$$\therefore\ L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{a\sin at-b\sin bt}{a^2-b^2}$$

Q73. Find the inverse Laplace transformation of $\left[\dfrac{s+3}{(s^2+6s+13)^2}\right]$.

Answer :

Given function is, $\dfrac{s+3}{(s^2+6s+13)^2}$

Then, $L^{-1}\left[\dfrac{s+3}{(s^2+6s+13)^2}\right] = L^{-1}\left[\dfrac{s+3}{[(s+3)^2+4][(s+3)^2+4]}\right]$

Let, $\bar{f}(s) = \dfrac{s+3}{(s+3)^2+2^2}$

$\Rightarrow \quad f(t) = L^{-1}\left[\dfrac{s+3}{(s+3)^2+2^2}\right] = e^{-3t}\cos 2t \qquad \left[\because L^{-1}\left(\dfrac{s+a}{(s+a)^2+b^2}\right) = (e^{-at}\cos bt)\right]$

And $g(s) = \dfrac{1}{(s+3)^2+2^2}$

$\Rightarrow \quad g(t) = L^{-1}\left(\dfrac{1}{(s+3)^2+2^2}\right) = e^{-3t}\dfrac{\sin 2t}{2} \qquad \left[\because L^{-1}\left(\dfrac{1}{(s+a)^2+b^2}\right) = e^{-at}\dfrac{\sin bt}{b}\right]$

By convolution theorem,

$$L^{-1}\left[\frac{s+3}{(s^2+6s+13)^2}\right] = \int_0^t f(u)g(t-u)\,du \qquad [\because L^{-1}[\bar{f}(s)\bar{g}(s)] = f(t) * g(t) = \int_0^t f(u)g(t-u)\,du]$$

$$= \int_0^\infty e^{-3u}\cos 2u . e^{-3(t-u)}\frac{\sin 2(t-u)}{2}\,du$$

$$= \frac{1}{2}\int_0^t e^{-3u}\cos 2u\, e^{-3t}\, e^{3u}\sin(2t-2u)\,du$$

$$= \frac{1}{2}e^{-3t}\int_0^t \cos 2u \;\sin(2t-2u)\,du$$

$$= \frac{1}{2}\frac{e^{-3t}}{2}\int_0^t 2\cos 2u\sin(2t-2u)\,du$$

$$= \frac{1}{4}e^{-3t}\int_0^t [\sin(2u+2t-2u)-\sin(2u-2t+2u)]du$$

$[\because 2\cos A\sin B = \sin(A+B) - \sin(A-B)]$

$$= \frac{e^{-3t}}{4}\int_0^t [\sin 2t - \sin(4u-2t)]du$$

$$= \frac{e^{-3t}}{4}\left[\int_0^t \sin 2t\,du - \int_0^t \sin(4u-2t)du\right]$$

$$= \frac{e^{-3t}}{4}\left[\sin 2t\int_0^t du - \left[\frac{-\cos(4u-2t)}{4}\right]_0^t\right]$$

$$= \frac{e^{-3t}}{4}[\sin 2t(u) + \left[\frac{\cos(4t-2t)}{4} - \frac{\cos(0-2t)}{4}\right]]$$

$$= \frac{e^{-3t}}{4}\left[t\sin 2t + \left[\frac{\cos 2t}{4} - \frac{\cos 2t}{4}\right]\right]$$

$$= \frac{e^{-3t}}{4}[t\sin 2t]$$

$$\therefore \; L^{-1}\left[\frac{s+3}{(s^2+6s+13)^2}\right] = \frac{e^{-3t}}{4}[t\sin 2t]$$

5.3 SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS USING LAPLACE TRANSFORMS

Q74. Using Laplace transform, solve (D^2 + 4D + 5)y = 5, given that y(0) = 0, y'(0) = 0.

Answer : Model Paper-1, Q17(b)

Given differential equation is,

$$(D^2 + 4D + 5)y = 5$$

$$y(0) = 0, \; y''(0) = 0$$

$$\Rightarrow \; D^2y + 4Dy + 5y = 5$$

$$\Rightarrow \; \frac{d^2y}{dx^2} + \frac{4dy}{dx} + 5y = 5 \qquad \left(\because \; D = \frac{dy}{dx}\right)$$

$$\Rightarrow \; y'' + 4y' + 5y = 5$$

Applying Laplace transform on both sides,

$$L\{y'' + 4y' + 5y\} = L\{5\}$$

$$\Rightarrow \; L\{y''\} + 4L\{y'\} + 5L\{y\} = L\{5\}$$

$$\Rightarrow \; s^2L\{y\} - sy(0) - y'(0) + 4\{sL\{y\} - y(0)\} + 5L\{y\} = \frac{5}{s}$$

$$\Rightarrow \; (s^2 + 4s + 5)L\{y\} - (s + 4)y(0) - y'(0) = \frac{5}{s}$$

$$\Rightarrow \; (s^2 + 4s + 5)L\{y\} - (s + 4)0 - 0 = \frac{5}{s}$$

$$\Rightarrow \; L\{y\} = \frac{5}{s(s^2+4s+5)}$$

The roots of $s^2 + 4s + 5$ are,

$$s = \frac{-4 \pm \sqrt{16-4\times 5}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = \frac{2(-2 \pm i)}{2} = -2 \pm i = -2 + i, -2 - i$$

$$\Rightarrow \; L\{y\} = \frac{5}{s(s+(2-i))(s+(2+i))} = \frac{A}{s} + \frac{B}{s+(2-i)} + \frac{C}{s+(2+i)} \quad \text{... (1)}$$

$$\Rightarrow \; \frac{5}{s(s+(2-i))(s+(2+i))} = \frac{A(s+(2-i))(s+(2+i)) + Bs(s+(2+i)) + C(s)(s+(2-i))}{s(s+(2-i))(s+(2+i))}$$

$$\Rightarrow \; 5 = A(s + (2-i))(s + (2+i)) + Bs(s + (2+i)) + C(s)(s + (2-i)) \quad \text{... (2)}$$

Substituting, $s = 0$ in equation (2),

$$5 = A(2 - i)(2 + i)$$

$$\Rightarrow \quad 5 = A[4 - (i^2)]$$

$$\Rightarrow \quad 5 = A[5]$$

$$\therefore A = 1$$

Substituting, $s = -(2 - i)$ in equation (2),

$$5 = B(-(2 - i))(-2 + i + 2 + i)$$

$$\Rightarrow \quad 5 = B(-2 + i)(2i)$$

$$\Rightarrow \quad 5 = B(-4i + 2i^2)$$

$$\Rightarrow \quad 5 = B(-4i - 2)$$

$$\therefore B = \frac{-5}{4i+2}$$

Substituting, $s = -(2 + i)$ in equation (2),

$$5 = C(-2 - i)(-2 - i + 2 - i)$$

$$\Rightarrow \quad 5 = C(-2 - i)(-2i)$$

$$\Rightarrow \quad 5 = C(2 + i)(2i)$$

$$\Rightarrow \quad 5 = C(4i + 2i^2)$$

$$\Rightarrow \quad 5 = C[4i - 2]$$

$$\therefore \quad C = \frac{5}{4i-2}$$

Substituting the corresponding values of A, B, C in equation (1),

$$\{L(y)\} = \frac{1}{s} - \frac{5}{(4i+2)}\left[\frac{1}{(s+(2-i))}\right] + \frac{5}{4i-2}\left[\frac{1}{(s+(2+i))}\right]$$

Applying inverse Laplace transform on both sides,

$$L^{-1}\{L(y)\} = L^{-1}\left[\frac{1}{s} - \frac{5}{(4i+2)}\left(\frac{1}{(s+(2-i))}\right) + \frac{5}{(4i-2)}\left(\frac{1}{s+(2+i)}\right)\right]$$

$$\Rightarrow \quad y = 1 - \frac{5}{4i+2}e^{-(2-i)t} + \frac{5}{4i-2}e^{-(2+i)t} \qquad \left(\because L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}\right)$$

$$\therefore \quad y = 1 - \frac{5}{4i+2}e^{-(2-i)t} + \frac{5}{4i-2}e^{-(2+i)t}$$

Q75. Solve ($D^4 - k^4$) y = 0 if y(0) = 1, y'(0) = 0, y''(0) = 0, y'''(0) = 0. Using Laplace transform method.

Answer :

Given differential equation is,

$$(D^4 - k^4)y = 0$$

Applying Laplace transform on both sides,

$$L[D^4 - k^4]y = 0$$

$$\Rightarrow \quad s^4 L[y] - s^3 y(0) - s^2 y'(0) - sy''(0) - y'''(0) - k^4 L[y] = 0$$

$$\Rightarrow \quad s^4 L[y] - s^3(1) - 0 - 0 - 0 - k^4 L[y] = 0$$

$\Rightarrow \quad s^4 L[y] - s^3 - k^4 L[y] = 0$

$\Rightarrow \quad s^4 L[y] - k^4 L[y] = s^3$

$\Rightarrow \quad [s^4 - k^4] L[y] = s^3$

$$\Rightarrow \quad L[y] = \frac{s^3}{s^4 - k^4} \quad \ldots (1)$$

Consider,

$$\frac{s^3}{s^4 - k^4} = \frac{s^3}{(s^2 + k^2)(s^2 - k^2)}$$

$$= \frac{As + B}{(s^2 + k^2)} + \frac{Cs + D}{(s^2 - k^2)} \quad \ldots (2)$$

$\Rightarrow \quad s^3 = (As + B)(s^2 - k^2) + (Cs + D)(s^2 + k^2)$

$\Rightarrow \quad s^3 = As^3 + Bs^2 - Ask^2 - Bk^2 + Cs^3 + Ds^2 + Csk^2 + Dk^2$

Comparing coefficients on both sides,

$A + C = 1 \quad \ldots (3)$

$\Rightarrow \quad -A + C = 0 \quad \ldots (4)$

$B + D = 0 \quad \ldots (5)$

$\Rightarrow \quad -B + D = 0 \quad \ldots (6)$

Solving equations (3) and (4),

$$A = C = \frac{1}{2}$$

Solving equations (5) and (6)

$B = D = 0$

Substituting the corresponding values in equation (2),

$$\frac{s^3}{s^4 - k^4} = \frac{\left(\frac{1}{2}\right)s + 0}{(s^2 + k^2)} + \frac{\frac{1}{2}s + 0}{s^2 - k^2}$$

$$\Rightarrow \quad \frac{s^3}{s^4 - k^4} = \frac{1}{2}\left[\frac{s}{s^2 + k^2} + \frac{s}{s^2 - k^2}\right] \quad \ldots (7)$$

Substituting equation (7) in equation (1),

$$L[y] = \frac{1}{2}\left[\frac{s}{s^2 + k^2} + \frac{s}{s^2 - k^2}\right]$$

Applying inverse Laplace transform on both sides,

$$y = L^{-1}\left[\frac{1}{2}\left[\frac{s}{s^2 + k^2} + \frac{s}{s^2 - k^2}\right]\right]$$

$$= \frac{1}{2}\left[L^{-1}\left[\frac{s}{s^2 + k^2}\right] + L^{-1}\left[\frac{s}{s^2 - k^2}\right]\right] \qquad \left[\begin{array}{l} \because L^{-1}\left[\frac{s}{s^2 + a^2}\right] = \cos at \\ L^{-1}\left[\frac{s}{s^2 - a^2}\right] = \cosh at \end{array}\right]$$

$$\therefore \; y = \frac{1}{2}[\cos kt + \cosh kt].$$

Q76. Solve (D^3 + D) x = 2 if x(0) = 3, x'(0) = 1, x''(0) = – 2, using Laplace transform method.

Answer : Model Paper-2, Q17(b)

Given differential equation is,

$$(D^3 + D)\, x = 2$$

$$\Rightarrow \quad \frac{d^3x}{dt^3} + \frac{dx}{dt} = 2$$

$$\Rightarrow \quad x''' + x' = 2$$

Applying Laplace transform on both sides,

$$L\{x''' + x'\} = L\{2\}$$

$$L\{x'''\} + L\{x'\} = \frac{2}{s}$$

$$s^3.X\{s\} - s^2\, x(0) - sx'(0) - x''(0) + s.X\{s\} - x(0) = \frac{2}{s}$$

Substituting the given initial conditions,

$$s^3.\, X\{s\} - s^2(3) - s(1) - (-2) + s.X\{s\} - 3 = \frac{2}{s}$$

$$\Rightarrow \quad s^3.\, X\{s\} - 3s^2 - s + 2 + s.\, X(s) - 3 = \frac{2}{s}$$

$$\Rightarrow \quad X\{s\}\,[s^3 + s\} - 3s^2 - s - 1 = \frac{2}{s}$$

$$\Rightarrow \quad X(s)\,.[s^3 + s] = \frac{2}{s} + 3s^2 + s + 1$$

$$\Rightarrow \quad X(s).(s^3 + s) = \frac{2 + 3s^3 + s^2 + s}{s}$$

$$\Rightarrow \quad X(s) = \frac{2 + 3s^3 + s^2 + s}{s(s^3 + s)}$$

$$\Rightarrow \quad X(s) = \frac{2 + 3s^3 + s^2 + s}{s^2(s^2 + 1)}$$

Applying partial fraction to above equation,

$$X(s) = \frac{2 + 3s^3 + s^2 + s}{s^2(s^2 + 1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{s^2 + 1} \quad \text{... (1)}$$

$$\Rightarrow \quad \frac{2 + 3s^3 + s^2 + s}{s^2(s^2 + 1)} = \frac{As(s^2 + 1) + B(s^2 + 1) + (Cs + D)s^2}{s^2(s^2 + 1)}$$

$$\Rightarrow \quad 2 + 3s^3 + s^2 + s = As\ (s^2 + 1) + B(s^2 + 1) + s^2(Cs + D)$$

$$\Rightarrow \quad 2 + 3s^3 + s^2 + s = As^3 + As + Bs^2 + B + Cs^3 + Ds^2$$

$$\Rightarrow \quad 2 + 3s^3 + s^2 + s = (A + C)\, s^3 + (B + D)\, s^2 + As + B.$$

Comparing coefficients on both sides,

$$A = 1, B = 2$$

$A + C = 3 \Rightarrow C = 3 - A$

$\Rightarrow \quad C = 3 - 1$

$\Rightarrow \quad C = 2$

$B + D = 1 \Rightarrow D = 1 - B$

$\Rightarrow \quad D = 1 - 2$

$\Rightarrow \quad D = -1$

Substituting the values of A, B, C and D in equation (1),

$$X(s) = \frac{1}{s} + \frac{2}{s^2} + \frac{2s-1}{s^2 + 1}$$

Applying inverse Laplace transform on both sides,

$$L^{-1}\{X(s)\} = L^{-1}\left\{\frac{1}{s}+\frac{2}{s^2}+\frac{2s-1}{s^2+1}\right\}$$

$$\Rightarrow \quad x(t) = L^{-1}\left\{\frac{1}{s}\right\}+2L^{-1}\left\{\frac{1}{s^2}\right\}+L^{-1}\left\{\frac{2s-1}{s^2+1}\right\}$$

$$=1+2t+2.L^{-1}\left\{\frac{s}{s^2+1}\right\}-L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= 1 + 2t + 2\cos t - \sin t$$

$$\therefore \quad x(t) = 1 + 2t + 2\cos t - \sin t.$$

Q77. Solve (D^2 + 9) y = cos2t, y(0) =1, $y\left(\frac{\pi}{2}\right)$ = –1 by using transform method.

Answer :

Given differential equation is,

$$(D^2 + 9)y = \cos 2t \quad \text{... (1)}$$

$$y(0) = 1,$$

$$y\left(\frac{\pi}{2}\right) = -1$$

$$\frac{d^2y}{dt^2}+9y = \cos 2t \qquad \left[\because D = \frac{d}{dt}\right]$$

Applying Laplace transform on both sides,

$$L\left\{\frac{d^2y}{dt^2}+9y\right\} = L\{\cos 2t\}$$

$$\Rightarrow \quad L\left\{\frac{d^2y}{dt^2}\right\}+9L\{y\} = L\{\cos 2t\}$$

$$\Rightarrow \quad \{s^2\bar{y}(s)-sy(0)-y'(0)\}+9\bar{y}(s) = \frac{s}{s^2+2^2}$$

$$\Rightarrow \quad \{s^2\bar{y}(s)-s(1)-q\}+9\bar{y}(s) = \frac{s}{s^2+4} \qquad \left[\because L\{\cos at\} = \frac{s}{s^2+a^2}\right]$$

$$\Rightarrow \quad \bar{y}(s)\{s^2+9\} - \frac{s}{s^2+4}+s+q$$

$$\Rightarrow \quad \bar{y}(s)\{s^2+9\} = \frac{s+(s+q)(s^2+4)}{s^2+4}$$

$$\Rightarrow \quad \bar{y}(s) = \frac{s}{(s^2+4)(s^2+9)}+\frac{s+q}{s^2+9} \quad \text{... (2)}$$

Applying partial fractions,

$$\frac{1}{(s^2+4)(s^2+9)} = \frac{As+D}{s^2+4}+\frac{Bs+C}{s^2+9} \quad \text{... (3)}$$

$$\Rightarrow \quad 1 = (As + D)(s^2 + 9) + (Bs + C)(s^2 + 4)$$

Comparing coefficients on both sides,

$$A + B = 0 \quad \text{... (4)}$$

$$D + C = 0 \quad \text{... (5)}$$

$$9A + 4B = 0 \quad \text{... (6)}$$

$$9D + 4C = 1 \quad \text{... (7)}$$

Solving equations (4) and (6),

$$A = 0, B = 0$$

Solving equations (5) and (7),

$$C = \frac{-1}{5}, D = \frac{1}{5}$$

Substituting the corresponding values of A, B, C and D in equation (3),

$$\frac{1}{(s^2+4)(s^2+9)} = \frac{\frac{1}{5}}{s^2+4} + \frac{\frac{-1}{5}}{s^2+9} \quad \text{... (8)}$$

Substituting equation (8) in equation (2),

$$\overline{y}(s) = s\left[\frac{\frac{1}{5}}{s^2+4} - \frac{\frac{1}{5}}{s^2+9}\right] + \frac{s}{s^2+9} + \frac{q}{s^2+9}$$

$$\Rightarrow \quad \overline{y}(s) = \frac{1}{5}\left[\frac{s}{s^2+4}\right] + \frac{s}{s^2+9}\left[1 - \frac{1}{5}\right] + \frac{q}{s^2+9}$$

$$\Rightarrow \quad \overline{y}(s) = \frac{1}{5}\left[\frac{s}{s^2+4}\right] + \frac{4}{5}\left[\frac{s}{s^2+9}\right] + \frac{q}{s^2+9}$$

Applying inverse Laplace transform on both sides above equation,

$$L^{-1}\{\overline{y}(s)\} = \frac{1}{5}L^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{4}{5}\left\{\frac{s}{s^2+9}\right\} + \frac{q}{s^2+9}$$

$$y(t) = \frac{1}{5}\cos(2t) + \frac{4}{5}\cos 3t + \frac{q}{3}\sin 3t \quad \text{... (9)}$$

Substituting $t = \frac{\pi}{2}$ in equation (9),

$$y\left(\frac{\pi}{2}\right) = \frac{1}{5}\cos\left(2.\frac{\pi}{2}\right) + \frac{4}{5}\cos\left(3.\frac{\pi}{2}\right) + \frac{q}{3}\sin\left(3.\frac{\pi}{2}\right)$$

$$\Rightarrow \quad -1 = \frac{1}{5}\cos(\pi) + \frac{4}{5}\cos\left(\frac{3\pi}{2}\right) + \frac{q}{3}\sin\left(\frac{3\pi}{2}\right)$$

$$\Rightarrow \quad -1 = \frac{-1}{5} + \frac{4}{5}(0) + \frac{q}{3}(-1)$$

$$\Rightarrow \quad -1 = \frac{-1}{5} - \frac{q}{3}$$

$$\Rightarrow \quad -1+\frac{1}{5} = -\frac{q}{3}$$

$$-\frac{4}{5} = -\frac{q}{3}$$

$$\therefore \quad q = \frac{12}{5}$$

Substituting the value of 'q' in equation (9),

$$y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{12}{5.3}\sin 3t$$

$$\therefore \; y(t) = \frac{1}{5}[\cos 2t + 4\cos 3t + 4\sin 3t]$$

Q78. Solve y'' – 3y' + 2y = 1 given that y(0) = 0, y'(0) = 1 by using Laplace transform method.

Answer :

Given differential equation is,

$$y'' - 3y' + 2y = 1 \qquad \text{... (1)}$$

And $y(0) = 0$, $y'(0) = 1$

Applying Laplace transform on both sides of equation (1),

$$L\{y'' - 3y' + 2y\} = L\{1\}$$

$$\Rightarrow \quad L\{y''\} - 3L\{y'\} + 2L\{y\} = \frac{1}{s}$$

$$\Rightarrow \quad s^2L(y) - sy(0) + s^0y'(0) - 3[sL\{(y) - s^0y(0)\} + 2L(y)] = \frac{1}{s}$$

$$\Rightarrow \quad s^2L(y) - s(0) - (1) - 3[sL(y) - 0] + 2L(y) = \frac{1}{s}$$

$$\Rightarrow \quad s^2L(y) - 1 - 3sL(y) + 2L(y) = \frac{1}{s}$$

$$\Rightarrow \quad L(y)[s^2 - 3s + 2] - 1 = \frac{1}{s}$$

$$\Rightarrow \quad L(y)[s^2 - 3s + 2] = \frac{1}{s} + 1$$

$$\Rightarrow \quad L(y) = \frac{1+s}{s(s^2-3s+2)}$$

$$= \frac{1+s}{s(s^2-2s-s+2)}$$

$$= \frac{1+s}{s(s(s-2)-1(s-2))}$$

$$\Rightarrow \quad L(y) = \frac{1+s}{s(s-1)(s-2)}$$

Taking partial fractions,

$$\frac{1+s}{s(s-1)(s-2)} = \frac{A}{s}+\frac{B}{s-1}+\frac{C}{s-2} \quad \text{... (2)}$$

$$\Rightarrow \quad \frac{1+s}{s(s-1)(s-2)} = \frac{A(s-1)(s-2)+B(s)(s-2)+C(s)(s-1)}{s(s-1)(s-2)}$$

$$\Rightarrow \quad 1+s = A(s-1)(s-2)+Bs(s-2)+C(s)(s-1) \quad \text{... (3)}$$

Substituting $s = 0$ in equation (3),

$$1+0 = A(0-1)(0-2)+B(0)(0-2)+C(0)(0-1)$$

$$\Rightarrow \quad A = \frac{1}{2}$$

Substituting $s = 1$ in equation (3),

$$1+1 = A(1-1)(1-2)+B(1)(1-2)+C(1)(1-1)$$

$$\Rightarrow \quad 2 = 0+(-B)+0$$

$$\Rightarrow \quad B = -2$$

Substituting $s = 2$ in equation (3),

$$1+2 = A(2-1)(2-2)+B(2)(2-2)+C(2)(2-1)$$

$$\Rightarrow \quad 3 = 0+0+2C$$

$$\Rightarrow \quad C = \frac{3}{2}$$

Substituting the corresponding values in equation (2),

$$L(y) = \frac{1+s}{s(s-1)(s-2)} = \frac{1}{2s}+\frac{(-2)}{s-1}+\frac{3}{2(s-2)}$$

$$y = L^{-1}\left\{\frac{1}{2s}-\frac{2}{s-1}+\frac{3}{2}\frac{1}{s-2}\right\}$$

$$= \frac{1}{2}L^{-1}\left\{\frac{1}{s}\right\}-2L^{-1}\left\{\frac{1}{s-1}\right\}+\frac{3}{2}L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= \frac{1}{2}(1)-2e^{t}+\frac{3}{2}e^{2t}$$

$$\therefore \quad y = \frac{1-4e^{t}+3e^{2t}}{2}$$

Q79. Solve the D.E (D^2 + 6D + 9) = sint if y(0) = 1 and y'(0) = 0 using Laplace transform.

Answer : **Model Paper-3, Q17(b)**

Given that,

$$(D^2+6D+9) = \sin t \quad \text{... (1)}$$

And $y(0) = 1$; $y'(0) = 0$

Equation (1) can be rewritten as,

$$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=\sin t \qquad \left[\because D=\frac{dy}{dx}\right]$$

$\Rightarrow \quad y''+6y'+9y=\sin t$ ... (2)

Applying Laplace transform on both sides of equation (2),

$L\{y''+6y'+9y\}=L\{\sin t\}$

$\Rightarrow \quad L\{y''\}+6L\{y'\}+9L\{y\}=L\{\sin t\}$

$\Rightarrow \quad \{s^2L(y)-s(y(0))-y'(0)\}+6\{s\,L(y)-y(0)\}+9L\{y\}=L\{\sin t\}$

$[\because L\{y^n\}=s^nL\{y\}-s^{n-1}y\{0\}-s^{n-2}y'(0)]$ ... (3)

Since $y(0)=1$ and $y'(0)=0$, equation (3) becomes,

$\{s^2L(y)-s(1)-0\}+6\{s\,L(y)-1\}+9L\{y\}=L\{\sin t\}$

$\Rightarrow \quad \{s^2L(y)-s\}+6\{s\,L(y)-1\}+9L\{y\}=L\{\sin t\}$

$\Rightarrow \quad s^2L(y)-s+6\,s\,L(y)-6+9L(y)=L\{\sin t\}$

$\Rightarrow \quad \{s^2+6s+9\}\,L(y)-s-6=\dfrac{1}{1+s^2} \qquad \left[\because L(\sin t)=\dfrac{1}{1+s^2}\right]$

$\Rightarrow \quad \{s^2+6s+9\}\,L(y)=\dfrac{1}{1+s^2}+s+6$

$\Rightarrow \quad L(y)=\dfrac{1}{(s^2+6s+9)(1+s^2)}+\dfrac{s+6}{(s^2+6s+9)}$ ... (4)

Applying partial fraction to the first term of equation (4),

$$\frac{1}{(s+3)^2(1+s^2)}=\frac{A}{(s+3)^2}+\frac{B}{s+3}+\frac{Cs+D}{s^2+1}$$

$\Rightarrow \quad 1=A(s^2+1)+B(s+3)(s^2+1)+(Cs+D)(s+3)^2$ ... (5)

Substituting $s=-3$ in equation (5),

$1=A((-3)^2+1)+0+0$

$\Rightarrow \quad 1=A(9+1)$

$\Rightarrow \quad 1=A\,(10)$

$\Rightarrow \quad A=\dfrac{1}{10}$

$\therefore\ A=\dfrac{1}{10}$

Equating the coefficients of s^3 on both sides, of equation (5),

$0=B+C$

$\Rightarrow \quad B=-C$ ... (6)

Equating the coefficients of s on both the sides of equation (5),

$0=B+9C+6D$ ... (7)

Equating the constant terms on both sides of equation (5),

$1=A+3B+9D$

$\Rightarrow \quad 1=\dfrac{1}{10}+3B+9D$

$$\Rightarrow \quad 3B + 9D = 1 - \frac{1}{10}$$

$$\Rightarrow \quad 3B + 9D = \frac{9}{10} \qquad \text{.... (8)}$$

Substituting $C = -B$ in equation (7),

$$-8B + 6D = 0 \qquad \text{... (9)}$$

Substituting equations (8) and (9),

$$\left(3B + 9D = \frac{9}{10}\right) \times 8$$

$$(-8B + 6D = 0) \times 3$$

$$24B + 72D = \frac{72}{10}$$

$$-24B + 18D = 0$$

$$90D = \frac{72}{10}$$

$$\Rightarrow \quad D = \frac{72}{10 \times 90}$$

$$\therefore D = \frac{2}{25}$$

From equation (9),

$$-8B + 6 \times \left(\frac{2}{25}\right) = 0$$

$$\Rightarrow \quad -8B + \frac{12}{25} = 0$$

$$\Rightarrow \quad 8B = \frac{12}{25}$$

$$\Rightarrow \quad B = \frac{12}{25 \times 8}$$

$$\Rightarrow \quad = \frac{3}{50}$$

$$\therefore B = \frac{3}{50}$$

From equation (6),

$$B + C = 0$$

$$\Rightarrow \quad \frac{3}{50} + C = 0$$

$$\Rightarrow \quad C = \frac{-3}{50}$$

$$\therefore C = \frac{-3}{50}$$

$$\Rightarrow \quad \frac{1}{(s+3)^2(s^2+1)} = \frac{1}{10}\left\{\frac{1}{(s+3)^2}\right\}+\frac{3}{50}\left\{\frac{1}{(s+3)}\right\}+\left(\frac{-3}{50}\right)\left\{\frac{s}{s^2+1}\right\}+\frac{2}{25}\left\{\frac{1}{s^2+1}\right\} \qquad \text{... (10)}$$

Consider second term of equation (4),

i.e., $$\frac{s+6}{(s+3)^2} = \frac{s+3}{(s+3)^2}+\frac{3}{(s+3)^2}$$

$$\frac{s+6}{(s+3)^2} = \frac{1}{s+3}+\frac{3}{(s+3)^2} \qquad \text{... (11)}$$

Combining the equations (10) and (11),

$$L(y) = \frac{1}{10}\left\{\frac{1}{(s+3)^2}\right\}+\frac{3}{50}\left\{\frac{1}{(s+3)}\right\}\frac{-3}{50}\left\{\frac{s}{s^2+1}\right\}+\frac{2}{25}\left\{\frac{1}{s^2+1}\right\}+\left\{\frac{1}{s+3}\right\}+3\left\{\frac{1}{(s+3)^2}\right\}$$

$$\Rightarrow \quad L(y) = \frac{31}{10}\left\{\frac{1}{(s+3)^2}\right\}+\frac{53}{50}\left\{\frac{1}{(s+3)}\right\}\frac{-3}{50}\left\{\frac{s}{s^2+1}\right\}+\frac{2}{25}\left\{\frac{1}{s^2+1}\right\}$$

Applying inverse Laplace transform on both sides,

$$L^{-1}(y) = \frac{31}{10}L^{-1}\left\{\frac{1}{(s+3)^2}\right\}+\frac{53}{50}L^{-1}\left\{\frac{1}{s+3}\right\}\frac{-3}{50}L^{-1}\left\{\frac{s}{s^2+1}\right\}+\frac{2}{25}L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{31}{10}te^{-3t}+\frac{53}{50}e^{-3t}-\frac{3}{50}\cos t+\frac{2}{25}\sin t$$

$$\therefore \quad y = \frac{31}{10}te^{-3t}+\frac{53}{50}e^{-3t}-\frac{3}{50}\cos t+\frac{2}{25}\sin t$$